



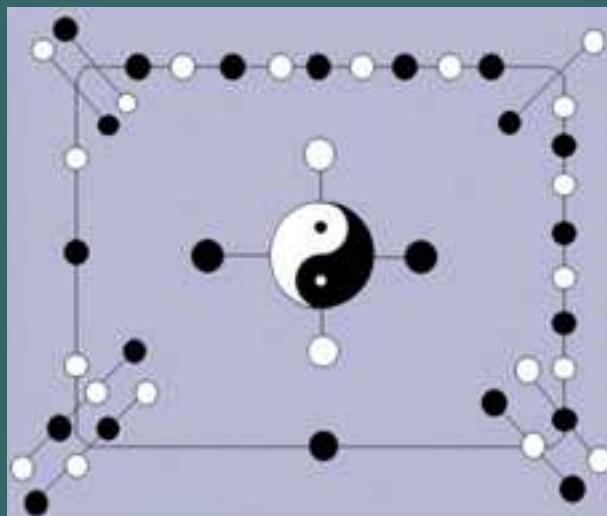
ISBN 978-1-59973-525-2

VOLUME 2, 2017

MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS, USA

June, 2017

Vol.2, 2017

ISBN 978-1-59973-525-2

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([www.mathcombin.com](http://www.mathcombin.com))

The Madis of Chinese Academy of Sciences and  
Academy of Mathematical Combinatorics & Applications, USA

June, 2017

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**Famous Words:**

*I want to bring out the secrets of nature and apply them for the happiness of man. I don't know of any better service to offer for the short time we are in the world.*

By Thomas Edison, an American inventor.

## A New Approach on the Striction Curves Belonging to Bertrandian Frenet Ruled Surfaces

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**Abstract:** In this study, we think six special ruled surfaces associated to the Bertrand curves pair  $\{\alpha, \alpha^*\}$ . We describe Bertrandian Frenet ruled surfaces and striction curves of these surfaces are expressed, as depending on the angle between the tangent vectors of the Bertrand curves pair  $\{\alpha, \alpha^*\}$ . Also, we examined the situation of the tangent vectors belonging to Striction curves of Frenet and Bertrandian Frenet ruled surfaces.

**Key Words:** Striction curves, Bertrand curves pair, ruled surfaces, Bertrandian Frenet ruled surface, Frenet ruled surface.

**AMS(2010):** 53A04, 53A05.

### §1. Introduction and Preliminaries

A surface is told to be ruled if it is generated by moving a direct line continuously in Euclidean space [8]. Ruled surfaces are one of the simplest objects in geometric modeling. The basis notions on ruled surfaces in are given in [4]. A ruled surface is a surface swept out by a straight line  $L$  moving along a curve  $\alpha(u) \in \mathbb{R}^3$ . The various positions of the generating line  $L$  are called the rulings of the surface. Such a surface always has a ruled parametrization

$$\varphi(u, v) = \alpha(u) + vX(u), \quad u \in I, \quad v \in \mathbb{R} \quad (1.1)$$

We call  $\alpha$  base curve and  $X$  the director curve, although  $X$  is usually pictured as a vector field on  $\alpha$  pointing along the line  $L$ . The striction point on a ruled surface is the foot of the common normal between two consecutive generators. The set of striction points defines the striction curve given by ([2])

$$c(s) = \alpha(s) - \frac{\langle \alpha', X' \rangle}{\langle X', X' \rangle} X. \quad (1.2)$$

Bertrand curves discovered by J. Bertrand in 1850 are one of the important and interesting

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<sup>1</sup>Received October 10, 2016, Accepted May 10, 2017.

topics of classical special curve theory. These well known properties of Bertrand curves in Euclidean 3-space was extended by L. R. Pears in [9]. The set, whose elements are frame vectors and curvatures of a curve  $\alpha$ , is called Frenet-Serret apparatus of the curves. Let Frenet vector fields be  $V_1(s), V_2(s), V_3(s)$  of  $\alpha$  and let the first and second curvatures of the curve  $\alpha(s)$  be  $k_1(s)$  and  $k_2(s)$ , respectively. The quantities  $\{V_1, V_2, V_3, k_1, k_2\}$  are collectively Frenet-Serret apparatus of the curves, ([3], [6]). The Frenet formulae are known as

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Let  $\alpha : I \rightarrow \mathbb{E}^3$  and  $\alpha^* : I \rightarrow \mathbb{E}^3$  be the  $C^2$ - class differentiable two curves and let  $V_1(s), V_2(s), V_3(s)$  and  $V_1^*(s), V_2^*(s), V_3^*(s)$  be the Frenet frames of the curves  $\alpha$  and  $\alpha^*$ , respectively. If the principal normal vector  $V_2$  of the curve  $\alpha$  is linearly dependent on the principal normal vector  $V_2^*$  of the curve  $\alpha^*$ , then the pair  $\{\alpha, \alpha^*\}$  are called Bertrand curves pair, [4], [7]. Also  $\alpha^*$  is called Bertrand mate. If the curve  $\alpha^*$  is Bertrand mate of  $\alpha$ , then we may write that

$$\alpha^*(s) = \alpha(s) + \lambda V_2(s). \quad (1.3)$$

If the curve  $\alpha^*$  is Bertrand mate  $\alpha(s)$ , then we have that  $\langle V_1^*(s), V_1(s) \rangle = \cos \theta = \text{constant}$ . The distance between corresponding points of the Bertrand curves pair in  $\mathbb{E}^3$  is constant.  $\alpha(s)$  is a Bertrand curve if and only if there exist nonzero real numbers  $\lambda$  and  $\beta$  such that constant  $\lambda k_1 + \beta k_2 = 1$  for any  $s \in I$ , [4], [7]. The relationship between Frenet apparatus belonging to  $\alpha$  and the Bertrand mate  $\alpha^*$  is as follows:

$$\begin{cases} V_1^* = \cos \theta V_1 + \sin \theta V_3 \\ V_2^* = V_2 \\ V_3^* = -\sin \theta V_1 + \cos \theta V_3. \end{cases} \quad (1.4)$$

$$k_1^* = \frac{\lambda k_1 - \sin^2 \theta}{\lambda(1 - \lambda k_1)}, \quad k_2^* = \frac{\sin^2 \theta}{\lambda^2 k_2}, \quad (1.5)$$

where  $\angle(V_1, V_1^*) = \theta$ , [10]. Due to this equations, we can write

$$\begin{aligned} \frac{k_1^*}{\eta^*} &= \frac{\lambda k_1 - \sin^2 \theta}{\lambda(1 - \lambda k_1) \left( \frac{(\lambda k_1 - \sin^2 \theta)^2}{\lambda^2(1 - \lambda k_1)^2} + \frac{\sin^4 \theta}{\lambda^4 k_2^2} \right)}, \\ \left( \frac{k_1^*}{\eta^*} \right)' &= \left[ \frac{(\lambda k_1 - \sin^2 \theta) \lambda^3 k_2^2 (1 - \lambda k_1)}{\left( (\lambda k_1 - \sin^2 \theta)^2 \lambda^2 k_2^2 + \sin^4 \theta (1 - \lambda k_1)^2 \right)} \right]' \frac{1}{k_2 \sqrt{\lambda^2 + \beta^2}}, \\ \frac{ds}{ds^*} &= \frac{1}{k_2 \sqrt{\lambda^2 + \beta^2}}, \end{aligned}$$

where  $\eta^* = k_1^{*2} + k_2^{*2}$ . Frenet ruled surface can be generated by the motion of a Frenet vector of

curve in  $\mathbb{E}^3$ . Tangent, Normal and Binormal ruled surfaces are collectively named Frenet ruled surfaces, [5].

**Theorem 1.1([5])** *Striction curves belonging to Frenet ruled surfaces are given in equations*

$$\begin{aligned} c_1(s) &= c_3(s) = \alpha(s) \\ c_2(s) &= \alpha(s) + \frac{k_1(s)}{k_2^2(s) + k_2^2(s)} V_2(s) \end{aligned} \quad (0.1)$$

**Theorem 1.2([5])** *Tangent vector fields  $T_1$ ,  $T_2$  and  $T_3$  of striction curves belonging to Frenet ruled surface are given by*

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{k_2^2}{\eta \|c'_2(s)\|} & \frac{(\frac{k_1}{\eta})'}{\|c'_2(s)\|} & \frac{k_1 k_2}{\eta \|c'_2(s)\|} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad (1.7)$$

where  $\eta = k_1^2 + k_2^2$ .

## §2. A New Approach on the Striction Curves Belonging to Bertrandian

### Frenet Ruled Surfaces

In this section first, we give Bertrandian tangent, Bertrandian normal and Bertrandian binormal Frenet ruled surfaces depending on the angle between the tangent vectors of the Bertrand curves pair. The later, we examined the situation of the tangent vectors belonging to Striction curves of Frenet and Bertrandian Frenet ruled surfaces.

**Definition 2.1** *Let Bertrand curves pair  $\{\alpha, \alpha^*\}$ . Bertrandian Frenet ruled surfaces which are depended on the angle between the tangent vectors of Bertrand curves pair are defined as follow, respectively*

$$\begin{aligned} \varphi_1^*(s, w_1) &= \alpha^*(s) + w_1 V_1^*(s) = \alpha + \lambda V_2 + w_1 (\cos \theta V_1 + \sin \theta V_3), \\ \varphi_2^*(s, w_2) &= \alpha^*(s) + w_2 V_2^*(s) = \alpha + (\lambda + w_2) V_2, \\ \varphi_3^*(s, w_3) &= \alpha^*(s) + w_3 V_3^*(s) = \alpha + \lambda V_2 + w_3 (-\sin \theta V_1 + \cos \theta V_3). \end{aligned} \quad (2.1)$$

**Theorem 2.2([11])** *Striction curves belonging to Bertrandian Frenet ruled surfaces depending on the angle between the tangent vectors of Bertrand curves pair can be expressed as follows,*

$$\begin{aligned}
c_1^* &= \alpha + \lambda V_2, \\
c_2^* &= \alpha + \lambda V_2 + \frac{(\lambda k_1 - \sin^2 \theta) V_2}{\lambda(1 - \lambda k_2) \left( \frac{(\lambda k_1 - \sin^2 \theta)^2}{\lambda^2(1 - \lambda k_2)^2} + \frac{\sin^4 \theta}{\lambda^4 k_2^2} \right)}, \\
c_3^* &= \alpha + \lambda V_2.
\end{aligned}$$

**Theorem 2.3([11])** *Tangent vector fields  $T_1^*$ ,  $T_2^*$ ,  $T_3^*$ , and  $T_4^*$  of striction curves belonging to Bertrandian Frenet ruled surfaces are given by*

$$\begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix} \quad (2.2)$$

where

$$\begin{aligned}
a^* &= \frac{k_2^{*2}}{\eta^* \|c_2^{*'}(s)\|} = \frac{\sin^4 \theta}{\lambda^4 k_2^2 \left( \frac{(\lambda k_1 - \sin^2 \theta)^2}{\lambda^2(1 - \lambda k_1)^2} + \frac{\sin^4 \theta}{\lambda^4 k_2^2} \right) \sqrt{\psi_1^2 + \psi_2^2 + \psi_3^2}}, \\
b^* &= \frac{\left( \frac{k_1^*}{\eta^*} \right)'}{\|c_2^{*'}(s)\|} = \frac{\left[ (\lambda k_1 - \sin^2 \theta) \lambda^{-1} (1 - \lambda k_1)^{-1} \left( \frac{(\lambda k_1 - \sin^2 \theta)^2}{\lambda^2(1 - \lambda k_1)^2} + \frac{\sin^4 \theta}{\lambda^4 k_2^2} \right)^{-1} \right]'}{k_2 \sqrt{(\lambda^2 + \beta^2)(\psi_1^2 + \psi_2^2 + \psi_3^2)}}, \\
c^* &= \frac{k_1^* k_2^*}{\eta^* \|c_2^{*'}(s)\|} = \frac{(\lambda k_1 - \sin^2 \theta) \sin^2 \theta}{\lambda^3 k_2 (1 - \lambda k_1) \left( \frac{(\lambda k_1 - \sin^2 \theta)^2}{\lambda^2(1 - \lambda k_1)^2} + \frac{\sin^4 \theta}{\lambda^4 k_2^2} \right) \sqrt{\psi_1^2 + \psi_2^2 + \psi_3^2}}.
\end{aligned}$$

$$\begin{aligned}
\psi_1 &= \left[ \frac{\sin^4 \theta \cos \theta}{\lambda^4 k_2^2 \left( \frac{(\lambda k_1 - \sin^2 \theta)^2}{\lambda^2(1 - \lambda k_1)^2} + \frac{\sin^4 \theta}{\lambda^4 k_2^2} \right)} - \frac{(\lambda k_1 - \sin^2 \theta) \sin^3 \theta}{\lambda^3 k_2 (1 - \lambda k_1) \left( \frac{(\lambda k_1 - \sin^2 \theta)^2}{\lambda^2(1 - \lambda k_1)^2} + \frac{\sin^4 \theta}{\lambda^4 k_2^2} \right)} \right], \\
\psi_2 &= \left[ \frac{\lambda k_1 - \sin^2 \theta}{\lambda (1 - \lambda k_1) \left( \frac{(\lambda k_1 - \sin^2 \theta)^2}{\lambda^2(1 - \lambda k_1)^2} + \frac{\sin^4 \theta}{\lambda^4 k_2^2} \right)} \right]' \frac{1}{k_2 \sqrt{\lambda + \beta}}, \\
\psi_3 &= \left[ \frac{\sin^5 \theta}{\lambda^4 k_2^2 \left( \frac{(\lambda k_1 - \sin^2 \theta)^2}{\lambda^2(1 - \lambda k_1)^2} + \frac{\sin^4 \theta}{\lambda^4 k_2^2} \right)} + \frac{(\lambda k_1 - \sin^2 \theta) \sin^2 \theta \cos \theta}{\lambda^3 k_2 (1 - \lambda k_1) \left( \frac{(\lambda k_1 - \sin^2 \theta)^2}{\lambda^2(1 - \lambda k_1)^2} + \frac{\sin^4 \theta}{\lambda^4 k_2^2} \right)} \right].
\end{aligned}$$

**Theorem 2.4** *The matrix of tangent vector fields on striction curves belonging to Frenet and*

Bertrandian Frenet ruled surfaces are given by

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \end{bmatrix}^T = \begin{bmatrix} \cos \theta & a^* \cos \theta - c^* \sin \theta & \cos \theta \\ X & a^* X + \frac{(\frac{k_1}{\eta})'}{\|c'_2(s)\|} b^* + c^* Y & X \\ \cos \theta & a^* \cos \theta - c^* \sin \theta & \cos \theta \end{bmatrix}$$

where  $X = \frac{k_2^2}{\eta \|c'_2(s)\|} \cos \theta + \frac{k_1 k_2}{\eta \|c'_2(s)\|} \sin \theta$ ,  $Y = -\frac{k_2^2}{\eta \|c'_2(s)\|} \sin \theta + \frac{k_1 k_2}{\eta \|c'_2(s)\|} \cos \theta$ .

*Proof* Let be  $[T] = [A][V]$  and  $[T^*] = [A^*][V^*]$ . By using the properties of the matrix, the following result is obtained.

$$\begin{aligned} [T][T^*]^T &= [A][V]([A^*][V^*])^T \\ &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{k_2^2}{\eta \|c'_2(s)\|} & \frac{(\frac{k_1}{\eta})'}{\|c'_2(s)\|} & \frac{k_1 k_2}{\eta \|c'_2(s)\|} \\ 1 & 0 & 0 \end{bmatrix} \left( \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}^T \right) \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ X & \frac{(\frac{k_1}{\eta})'}{\|c'_2(s)\|} & Y \\ \cos \theta & 0 & -\sin \theta \end{bmatrix} \begin{bmatrix} 1 & a^* & 1 \\ 0 & b^* & 0 \\ 0 & c^* & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & a^* \cos \theta - c^* \sin \theta & \cos \theta \\ X & a^* X + \frac{(\frac{k_1}{\eta})'}{\|c'_2(s)\|} b^* + c^* Y & X \\ \cos \theta & a^* \cos \theta - c^* \sin \theta & \cos \theta \end{bmatrix} \quad \square \end{aligned}$$

**Corollary 2.1** (i) If the Bertrand mate is helix curve, then tangent vector fields of striction curves belonging to Tangent and Bertrandian normal ruled surfaces can be orthogonal;

(ii) If the Bertrand mate is helix curve, then tangent vector fields of striction curves belonging to Binormal and Bertrandian normal ruled surfaces can be orthogonal.

*Proof* i) If inner product of  $T_1$  and  $T_2^*$  is zero, then  $T_1$  and  $T_2^*$  are orthogonal vectors.

$$\begin{aligned} \langle T_1, T_2^* \rangle = 0 \Rightarrow \langle T_1, T_2^* \rangle &= a^* \cos \theta - c^* \sin \theta = 0 \implies \frac{a^*}{c^*} = \frac{\sin \theta}{\cos \theta} = \tan \theta \\ \frac{k_2^*}{k_1^*} &= \tan \theta = \text{constant} \end{aligned}$$

this completes the proof.

For (ii), since  $T_1$  and  $T_3$  are equivalent vectors, then the proof is clear.  $\square$

**Corollary 2.2** (i) If Bertrand curve is helix curve, then tangent vector fields of striction curves belonging to Normal and Bertrandian tangent ruled surfaces can be orthogonal;

(ii) If the Bertrand curve is helix curve, then tangent vector fields of striction curves belonging to Normal and Bertrandian binormal ruled surfaces can be orthogonal.

*Proof* (i) If inner product of  $T_2$  and  $T_1^*$  is zero, then  $T_2$  and  $T_1^*$  are orthogonal vectors.

$$\begin{aligned}\langle T_2, T_1^* \rangle = 0 \Rightarrow \langle T_2, T_1^* \rangle = X &= \frac{k_2^2}{\eta \|c'_2(s)\|} \cos \theta + \frac{k_1 k_2}{\eta \|c'_2(s)\|} \sin \theta = 0 \\ \Rightarrow \frac{k_2}{k_1} &= -\tan \theta = \text{constant.}\end{aligned}$$

This completes the proof.

For (ii), since  $T_1^*$  and  $T_3^*$  are equivalent vectors, the proof is clear.  $\square$

**Corollary 2.3** *Tangent vector fields of striction curves belonging to Normal and Bertrandian normal ruled surfaces have orthogonal under the condition*

$$\left[ \frac{(\lambda k_1 - \sin^2 \theta) \lambda^3 k_2^2 (1 - \lambda k_1)}{\left( (\lambda k_1 - \sin^2 \theta)^2 \lambda^2 k_2^2 + \sin^4 \theta (1 - \lambda k_1)^2 \right)} \right]' = \frac{\frac{\sin^2 \theta}{\lambda^2 k_2} \left( \frac{\sin^2 \theta}{\lambda^2 k_2} X + \frac{\lambda k_1 - \sin^2 \theta}{\lambda (1 - \lambda k_1)} Y \right)}{-\frac{(\frac{k_1}{\eta})'}{\|c'_2(s)\|} \left( \frac{(\lambda k_1 - \sin^2 \theta)^2}{\lambda^2 (1 - \lambda k_1)^2} + \frac{\sin^4 \theta}{\lambda^4 k_2^2} \right)}.$$

$$\text{Proof } \langle T_2, T_2^* \rangle = 0 \Rightarrow \langle T_2, T_2^* \rangle = a^* X + \frac{(\frac{k_1}{\eta})'}{\|c'_2(s)\|} b^* + c^* Y = 0$$

$$\begin{aligned}a^* X + \frac{(\frac{k_1}{\eta})'}{\|c'_2(s)\|} b^* + c^* Y &= 0 \\ k_2^{*2} X + \frac{(\frac{k_1}{\eta})'}{\|c'_2(s)\|} \left( \frac{k_1^*}{\eta^*} \right)' \eta^* + k_1^* k_2^* Y &= 0 \\ k_2^* (k_2^* X + k_1^* Y) &= -\frac{(\frac{k_1}{\eta})'}{\|c'_2(s)\|} \left( \frac{k_1^*}{\eta^*} \right)' \eta^* \\ \left( \frac{k_1^*}{\eta^*} \right)' &= \frac{k_2^* (k_2^* X + k_1^* Y)}{-\frac{(\frac{k_1}{\eta})'}{\|c'_2(s)\|} \eta^*} \\ \left[ \frac{(\lambda k_1 - \sin^2 \theta) \lambda^3 k_2^2 (1 - \lambda k_1)}{\left( (\lambda k_1 - \sin^2 \theta)^2 \lambda^2 k_2^2 + \sin^4 \theta (1 - \lambda k_1)^2 \right)} \right]' &= \frac{\frac{\sin^2 \theta}{\lambda^2 k_2} \left( \frac{\sin^2 \theta}{\lambda^2 k_2} X + \frac{\lambda k_1 - \sin^2 \theta}{\lambda (1 - \lambda k_1)} Y \right)}{-\frac{(\frac{k_1}{\eta})'}{\|c'_2(s)\|} \left( \frac{(\lambda k_1 - \sin^2 \theta)^2}{\lambda^2 (1 - \lambda k_1)^2} + \frac{\sin^4 \theta}{\lambda^4 k_2^2} \right)}.\end{aligned}$$

This completes the proof.  $\square$

**Corollary 2.4** *Four pairs of tangent vectors fields of Frenet ruled surface belonging to Bertrand pair  $\{\alpha, \alpha^*\}$  are equal.*

*Proof* From the equations (1.7) and (2.2)

$$\langle T_1, T_1^* \rangle = \langle T_1, T_3^* \rangle = \langle T_3, T_1^* \rangle = \langle T_3, T_3^* \rangle = \langle V_1, \cos \theta V_1 + \sin \theta V_3 \rangle = \cos \theta. \quad \square$$

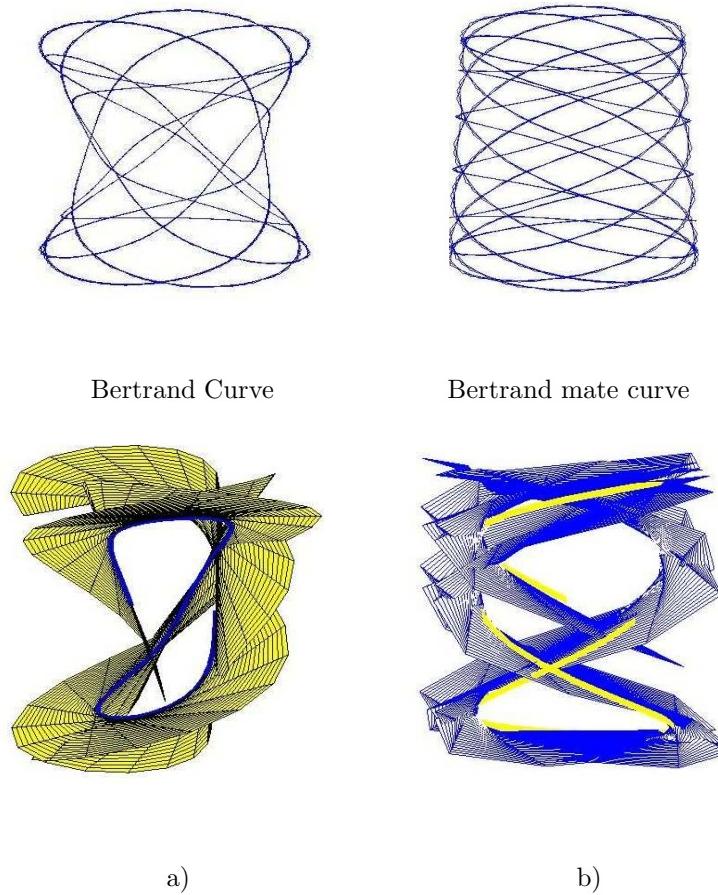
**Example 2.1** Let us consider the following Bertrand curve  $\alpha$  and Bertrand mate  $\alpha^*$ , respec-

tively.

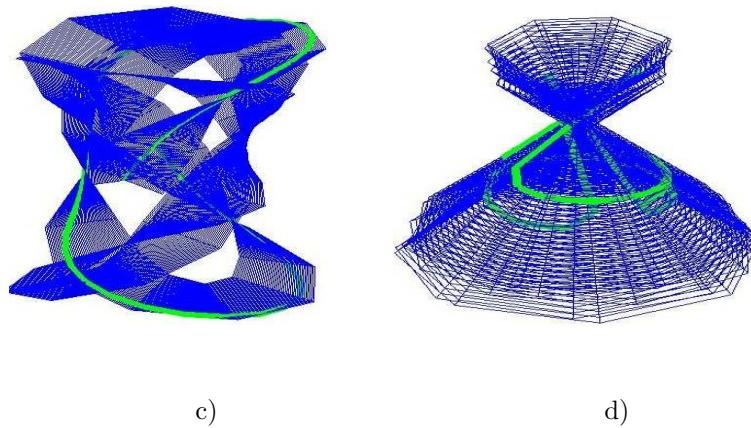
$$\begin{aligned}\alpha(s) &= \left( \frac{9}{208} \sin 16s - \frac{1}{117} \sin 36s, -\frac{9}{208} \cos 16s + \frac{1}{117} \cos 36s, \frac{6}{65} \sin 10s \right), \\ \alpha^*(s) &= \left( \frac{9}{208} \sin 16s - \frac{1}{117} \sin 36s + \frac{12}{13} \lambda \cos(26s), -\frac{9}{208} \cos 16s + \frac{1}{117} \cos 36s \right. \\ &\quad \left. + \frac{12}{13} \lambda \sin 26s, \frac{6}{65} \sin 10s - \frac{5}{13} \lambda \right), [?].\end{aligned}$$

Striction curves of the ruled surface formed by Frenet vector of this curves is as follows:

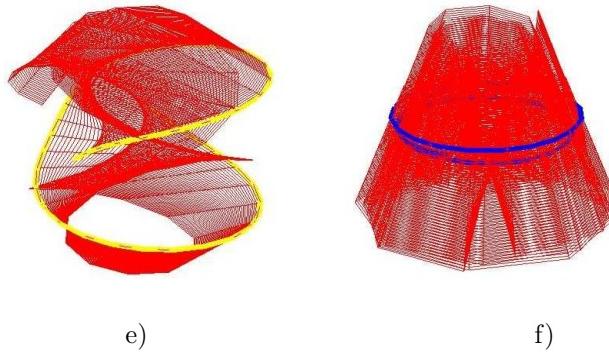
$$\left\{ \begin{aligned}c_1(s) &= \left( \frac{9}{208} \sin 16s - \frac{1}{117} \sin 36s, -\frac{9}{208} \cos 16s + \frac{1}{117} \cos 36s, \frac{6}{65} \sin 10s \right), \\ c_2(s) &= \left( \frac{9}{208} \sin 16s - \frac{1}{117} \sin 36s + \frac{1}{24} \sin 10s \left( \frac{9}{13} \cos 16s - \frac{4}{13} \cos 36s \right), \right. \\ &\quad \left. -\frac{9}{208} \cos 16s + \frac{1}{117} \cos 36s + \frac{1}{24} \sin 10s \left( \frac{9}{13} \sin 16s - \frac{4}{13} \sin 36s \right), \right. \\ &\quad \left. \frac{6}{65} \sin 10s + \frac{1}{26} \sin 10s \cos 10s \right), \\ c_3(s) &= \left( \frac{9}{208} \sin 16s - \frac{1}{117} \sin 36s, -\frac{9}{208} \cos 16s + \frac{1}{117} \cos 36s, \frac{6}{65} \sin 10s \right) \\ c_1^*(s) &= \left( \frac{9}{208} \sin 16s - \frac{1}{117} \sin 36s + \frac{12}{13} \lambda \cos(26s), -\frac{9}{208} \cos 16s + \frac{1}{117} \cos 36s \right. \\ &\quad \left. + \frac{12}{13} \lambda \sin 26s, \frac{6}{65} \sin 10s - \frac{5}{13} \lambda \right), \\ c_2^*(s) &= \left[ \left( \frac{9}{208} \sin 16s - \frac{1}{117} \sin 36s + \frac{12}{13} \lambda \cos 26s \right) \cos^2 10s + \frac{1}{14976} \sin 10s \cos 26s \right. \\ &\quad \left. - \frac{1}{624} \lambda \cos 26s, \cos^2 10s \left( -\frac{9}{208} \cos 16s + \frac{1}{117} \cos 36s + \frac{12}{13} \lambda \sin 26s \right) \right. \\ &\quad \left. + \frac{1}{14976} \sin 10s \sin 26s - \frac{1}{624} \lambda \sin 26s, \cos^2 10s \left( \frac{6}{65} \sin 10s - \frac{5}{13} \lambda \right) \right. \\ &\quad \left. - \frac{5}{179712} \sin 10s + \frac{5}{7488} \lambda \right] \cos^{-2} 10s, \\ c_3^*(s) &= \left( \frac{9}{208} \sin 16s - \frac{1}{117} \sin 36s + \frac{12}{13} \lambda \cos(26s), -\frac{9}{208} \cos 16s + \frac{1}{117} \cos 36s \right. \\ &\quad \left. + \frac{12}{13} \lambda \sin 26s, \frac{6}{65} \sin 10s - \frac{5}{13} \lambda \right),\end{aligned}\right.$$



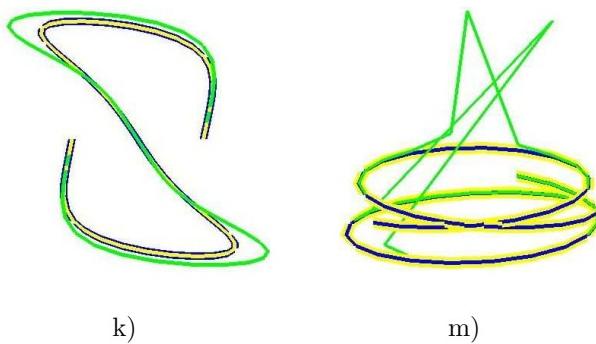
**Figure 1.** The figures a) and b) show, respectively, Tangent ruled surface and Bertrandian Tangent ruled surface. Blue and yellow colors are striction curves on these surfaces, respectively.



**Figure 2.** The figures c) and d) show, respectively, Normal ruled surface and Bertrandian Normal ruled surface. Green colors are striction curves on these surfaces, respectively.



**Figure 3.** The figures e) and f) show, respectively, Binormal ruled surface and Bertrandian Binormal ruled surface. Yellow and blue colors are striction curves on these surfaces, respectively.



**Figure 4.** The figures k) and m) show Striction curves of Frenet ruled surfaces and Bertrandian Frenet ruled surfaces.

These figures are drawn with Mapple program for  $\lambda = 1$ .

### Acknowledgement

This work was supported by BAP (The Scientific Research Projects Coordination Unit), Ordu University.

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## Mathematical Combinatorics with Natural Reality

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**Abstract:** There are 2 contradictory views on our world, i.e., continuous or discrete, which results in that only partially reality of a thing  $T$  can be understood by one of continuous or discrete mathematics because of the universality of contradiction and the connection of things in the nature, just as the philosophical meaning in the story of the blind men with an elephant. Holding on the reality of natural things motivates the combination of continuous mathematics with that of discrete, i.e., an envelope theory called *mathematical combinatorics* which extends classical mathematics over topological graphs because a thing is nothing else but a multiverse over a spacial structure of graphs with conservation laws hold on its vertices. Such a mathematical object is said to be an *action flow*. The main purpose of this survey is to introduce the powerful role of action flows, or mathematics over graphs with applications to physics, biology and other sciences, such as those of  $G$ -solution of non-solvable algebraic or differential equations, Banach or Hilbert  $\vec{G}$ -flow spaces with multiverse, multiverse on equations,  $\dots$  and with applications to complex systems, for examples, the understanding of particles, spacetime and biology. All of these make it clear that holding on the reality of things by classical mathematics is only on the coherent behaviors of things for its homogenous without contradictions, but the mathematics over graphs  $G$  is applicable for contradictory systems, i.e., complex systems because contradiction is universal only in eyes of human beings but not the nature of a thing itself.

**Key Words:** Graph, Banach space, Smarandache multispace,  $\vec{G}$ -flow, observation, natural reality, complex system, non-solvable equation, mathematical combinatorics.

**AMS(2010):** 03A10,05C15,20A05, 34A26,35A01,51A05,51D20,53A35.

### §1. Introduction

Generally, the reality of a thing  $T$  is its state of existed, exists, or will exist in the world, whether or not they are observable or comprehensible by human beings. However, the recognized reality maybe very different from that of the truth because it depends on the way of the observer and his world view is continuous or discrete, i.e., view the behavior of thing  $T$  a continuous function

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<sup>1</sup>Reported at the 2017 Spring International Conference on Applied and Engineering Mathematics, April 18-20, 2017, Chengdu, P.R.China.

<sup>2</sup>Received October 18, 2016, Accepted May 12, 2017.

$f$ , or an infinite or finite sequence  $x_1, x_2, \dots, x_n$  with  $n \geq 1$  on time  $t$ .

*Is our world continuous or discrete?* Certainly not because there exist both continuous or discrete things in the eyes of human beings. For example, all apples on a tree is discrete but the moving of a car on the road is continuous, such as those figures (a) and (b) shown in Fig.1.



(a)



(b)

**Fig.1**

And historically, holding on the behavior of things mutually develops the continuous and discrete mathematics, i.e., research a discrete (continuous) question by that of continuous (discrete) mathematical methods. For example, let  $x, y$  be the populations in a self-system of cats and rats, such as Tom and Jerry shown in Fig.2,

**Fig.2**

then they were continuously characterized by Lotka-Volterra with differential equations ([4])

$$\begin{cases} \dot{x} = x(\lambda - by), \\ \dot{y} = y(-\mu - cx). \end{cases} \quad (1.1)$$

Similarly, all numerical calculations by computer for continuous questions are carried out by discrete methods because algorithms language recognized by computer is essentially discrete. Such a typical example is the movies by discrete images for a continuous motion shown in Fig.3. Thus, the reality of things needs the combination of the continuous mathematics with that of the discrete.

**Fig.3**

Physically, the behavior of things  $T$  is usually characterized by differential equation

$$\mathcal{F}(t, x_1, x_2, x_3, \psi_t, \psi_{x_1}, \psi_{x_2}, \dots, \psi_{x_1 x_2}, \dots) = 0 \quad (1.2)$$

established on observed characters of  $\mu_1, \mu_2, \dots, \mu_n$  for its state function  $\psi(t, x)$  in reference system  $\mathbb{R}^3$  by Newtonian and  $\mathbb{R}^4$  by Einstein ([2]).

**Fig.4**

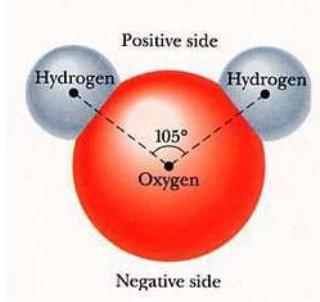
Usually, these physical phenomena of a thing is complex, and hybrid with other things. *Is the reality of particle  $P$  all solutions of that equation (1.2) in general?* Certainly not because the equation (1.2) only characterizes the behavior of  $P$  on some characters of  $\mu_1, \mu_2, \dots, \mu_n$  at time  $t$  abstractly, not the whole in philosophy. For example, the behavior of a particle is characterized by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + U\psi \quad (1.3)$$

in quantum mechanics ([24]) but observation shows it in two or more possible states of being, i.e., superposition such as the asking question of Schrödinger for the alive or dead of the cat in the box with poison switch shown in Fig.4. We can not even say which solution of the Schrödinger equation (1.3) is the particle because each solution is only for one determined state.

Furthermore, *can we conclude the equation (1.2) is absolutely right for a particle  $P$ ?* Certainly not also because the dynamic equation (1.2) is always established with an additional

assumption, i.e., the geometry on a particle  $P$  is a point in classical mechanics or a field in quantum mechanics and dependent on the observer is out or in the particle. For example, a water molecule  $H_2O$  consists of 2 hydrogen atoms and 1 oxygen atom such as those shown in Fig.5. If an observer receives information on the behaviors of hydrogen or oxygen atom but stands out of the water molecule  $H_2O$  by viewing it a geometrical point, then he can only receives coherent information on atoms  $H$  and  $O$  with the water molecule  $H_2O$ .



**Fig.5**

But if he enters the interior of the molecule, he will view a different sceneries for atoms  $H$  and  $O$ , which are respectively called *out-observation* and *in-observation*, and establishes equation (1.3) on  $H_2O$  or 3 dynamic equations

$$\left\{ \begin{array}{l} -i\hbar \frac{\partial \psi_O}{\partial t} = \frac{\hbar^2}{2m_O} \nabla^2 \psi_O - V(x)\psi_O \\ -i\hbar \frac{\partial \psi_{H_1}}{\partial t} = \frac{\hbar^2}{2m_{H_1}} \nabla^2 \psi_{H_1} - V(x)\psi_{H_1} \\ -i\hbar \frac{\partial \psi_{H_2}}{\partial t} = \frac{\hbar^2}{2m_{H_2}} \nabla^2 \psi_{H_2} - V(x)\psi_{H_2} \end{array} \right. \quad (1.4)$$

on atoms  $H$  and  $O$ . Which is the right model on  $H_2O$ , the (1.3) or (1.4) dynamic equations? The answer is not easy because the equation model (1.3) can only characterize those of coherent behavior of atoms  $H$  and  $O$  in  $H_2O$ , but equations (1.4) have no solutions, i.e., non-solvable in mathematics ([17]).

The main purpose of this survey is to clarify that the reality of a thing  $T$  should be a contradictory system in one's eyes, or multiverse with non-solvable systems of equations in geometry, conclude that they essentially describe its nature, which results in mathematical combinatorics, i.e., mathematics over graphs in space, and show its powerful role to mathematics with applications to elementary particles, gravitational field and other sciences, such as those of extended Banach or Hilbert  $\vec{G}$ -flow spaces, geometry on non-solvable systems of solvable differential equations, ··· with applications to the understanding of particles, population biology and other sciences.

For terminologies and notations not mentioned here, we follow references [1] for mechanics, [4] for biological mathematics, [8] for combinatorial geometry, [23]-[24] for elementary particles, and [25] for Smarandache systems and multispaces, and all phenomenons discussed in this paper are assumed to be true in the nature.

## §2. Contradiction, a By-product of Non-complete Recognizing

A philosophical proposition following clarifies the fundamental relation between the reality and the reality understood by classical mathematics, which is clear but few peoples noted in the past.

**Proposition 2.1** *Let  $\mathcal{R}$  and  $\mathcal{MR}$  be respectively the sets of reality and the reality known by classical mathematics on things. Then*

$$\mathcal{MR} \subset \mathcal{R} \quad \text{and} \quad \mathcal{MR} \neq \mathcal{R}. \quad (2.1)$$

*Proof* Notice that classical mathematical systems are homogenous without contradictions, i.e. a compatible one in logic but contradictions exist everywhere in philosophy. Thus, the reality known by classical mathematics on things can be only a subset of the reality set, i.e., the relation (2.1)

$$\mathcal{MR} \subset \mathcal{R} \quad \text{and} \quad \mathcal{MR} \neq \mathcal{R}. \quad \square$$

Although Proposition 2.1 is simple but it implies that for holding on reality of things, an envelope theory on classical mathematics, i.e., mathematics including contradictions is needed to establish for human beings.

### 2.1 Thinking Models

Let us discuss 3 thinking models on complex systems following.

**T1. The Blind Men with an Elephant.** This is a famous story in Buddhism which implies the entire consisting of its parts but we always hold on parts. In this story, there are six blind men were asked to determine what an elephant looked like by feeling different parts of an elephant's body. The man touched the elephant's leg, tail, trunk, ear, belly or tusk respectively claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, such as those shown in Fig.6.

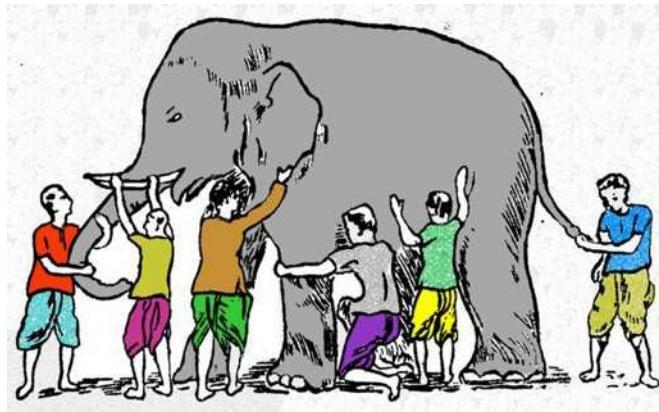


Fig.6

Each of these blind men insisted on his own's right, not accepted others, and then entered into an endless argument. *All of you are right!* A wise man explains to them: *why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said.* Hence, the wise man told these blind man that an elephant seemingly looked

$$\begin{aligned} \text{An elephant} = & \{4 \text{ pillars}\} \bigcup \{1 \text{ rope}\} \bigcup \{1 \text{ tree branch}\} \\ & \bigcup \{2 \text{ hand fans}\} \bigcup \{1 \text{ wall}\} \bigcup \{1 \text{ solid pipe}\} \end{aligned} \quad (2.2)$$

*What is the implication of this story for human beings?* It lies in the situation that human beings understand things in the world is analogous to these blind men. Usually, a thing  $T$  is understood by its known characters at one by one time and known gradually. For example, let  $\mu_1, \mu_2, \dots, \mu_n$  be known and  $\nu_i, i \geq 1$  unknown characters on a thing  $T$  at time  $t$ . Then,  $T$  is understood by

$$T = \left( \bigcup_{i=1}^n \{\mu_i\} \right) \bigcup \left( \bigcup_{k \geq 1} \{\nu_k\} \right) \quad (2.3)$$

in logic and with an approximation  $T^\circ = \bigcup_{i=1}^n \{\mu_i\}$  at time  $t$ . The equation (2.3) is called the *Smarandache multispace* ([8], [25]), a combination of discrete characters for understanding a thing  $T$ .

**T2. Everett's Multiverse on Superposition.** The multiverse interpretation by H.Everett [3] on wave function of equation (1.3) in 1957 answered the superposition of particles in machinery. By an assumption that the wave function of an observer would be interacted with a superposed object, he concluded different worlds in different quantum system obeying equation (1.3) and the superposition of a particle be liked those separate arms of a 2-branching universe ([16], [17]) such as those shown in Fig.7,



Fig.7

which revolutionary changed an ambiguous interpretation in quantum mechanics before him, i.e., an observer will cause the wave function to collapse randomly into one of the alternatives with all others disappearing. Everett's multiverse interpretation on the superposition of particle is in fact alluded in thinking model  $T1$ , i.e., the story of blind men with an elephant because

if one views each of these pillar, rope, tree branch, hand fan, wall and solid pipe by these blind men feeling on different parts of the elephant to be different spaces, then the looks of an elephant of the wise man told these blind men (2.2) is nothing else but an Everett's multiverse.

**T3. Quarks Model.** The divisibility of matter initiates human beings to search elementary constituting cells of matter, i.e., elementary particles such as those of quarks, leptons with interaction quanta including photons and other particles of mediated interactions, also with those of their antiparticles at present ([23], [24]), and unmatters between a matter and its antimatter which is partially consisted of matter but others antimatter ([26], [27]). For example, a baryon is predominantly formed by three quarks, and a meson is mainly composed of a quark and an antiquark in the models of Sakata, or Gell-Mann and Ne'eman, such as those shown in Fig.8, where there is also a particle composed of 5 quarks.

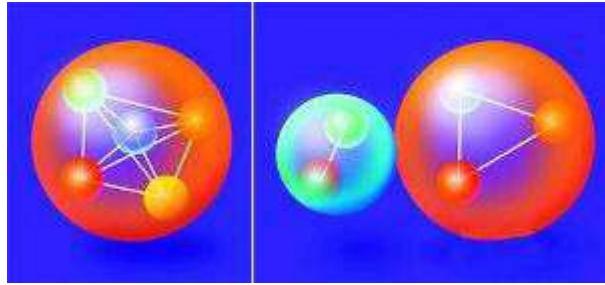


Fig.8

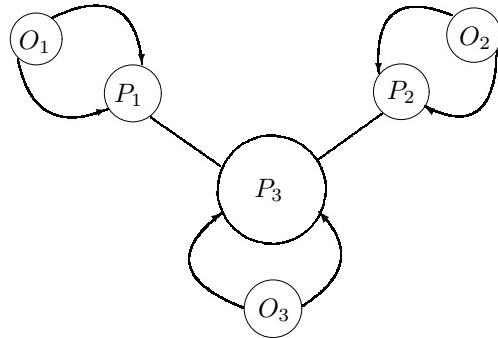
However, a free quark was never found in experiments. We can not even conclude the Schrödinger equations (1.3) is the right equation on quarks. But *why is it believed without a shadow of doubt that the dynamical equation of elementary particles such as those of quarks, leptons with interaction quanta is (1.3) in physics?* The reason is because that all observations come from a macro viewpoint, the human beings, not the quarks, and which can only lead to coherent behaviors, not the individuals. In mathematics, it is just an equation on those of particles viewed abstractly to be a geometrical point or an independent field from a macroscopic point, which results in physicists assuming the internal structures mechanically for understanding behaviors of particles, such as those shown in Fig.8. However, such an assumption is a little ambiguous in logic, i.e., we can not even distinguish who is the geometrical point or the field, the particle or its quark.

## 2.2 Contradiction Originated in Non-complete Recognizing

If we completely understand a thing  $T$ , i.e.,  $T = T^o$  in formula (2.3) at time  $t$ , there are no contradiction on  $T$ . However, this is nearly impossible for human beings, concluded in the first chapter of *TAO TEH KING* written by *Lao Zi*, a famous ideologist in China, i.e., “Name named is not the eternal; the without is the nature and naming the origin of things”, which also implies the universality of contradiction and a generalization of equation (2.1).

Certainly, the looks (2.2) of the wise man on the elephant is a complete recognizing but these of the blind men is not. However, *which is the right way of recognizing?* The answer

depends on the standing view of observer. The observation of these blind men on the elephant are a microscopic or in-observing but the wise man is macroscopic or out-observing. If one needs only for the macroscopic of an elephant, the wise man is right, but for the microscopic, these blind men are right on the different parts of the elephant. For understanding the reality of a thing  $T$ , we need the complete by individual recognizing, i.e., the whole by its parts. Such an observing is called a *parallel observing* ([17]) for avoiding the defect that each observer can only observe one behavior of a thing, such as those shown in Fig.9 on the water molecule  $H_2O$  with 3 observers.



**Fig.9**

Thus, the looks of the wise man on an elephant is a collection of parallel observing by these 6 blind men and finally results in the recognizing (2.2), and also the Everett's multiverse interpretation on the superposition, the models of Sakata, or Gell-Mann and Ne'eman on particles. This also concludes that multiverse exists everywhere if we observing a thing  $T$  by in-observation, not only those levels of  $I - IV$  classified by Max Tegmark in [28].

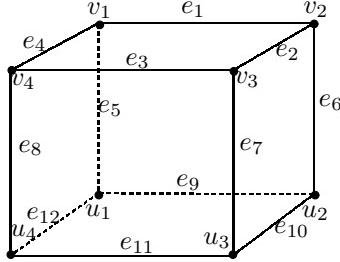
However, these equations (1.2) established on parallel observing datum of multiverse, for instance the equations (1.4) on 2 hydrogen atoms and 1 oxygen atom ([17]), and generally, differential equations (1.2) on population biology with more than 3 species are generally non-solvable. Then, *how to understand the reality of a thing  $T$  by mathematics holding with an equality  $\mathcal{M}\mathcal{R} = \mathcal{R}$ ?* The best answer on this question is the combination of continuous mathematics with that of the discrete, i.e., turn these non-mathematics in the classical to mathematics by a combinatorial manner ([13]), i.e., *mathematical combinatorics*, which is the appropriated way for understanding the reality because all things are in contradiction.

### §3. Mathematical Combinatorics

#### 3.1 Labeled Graphs

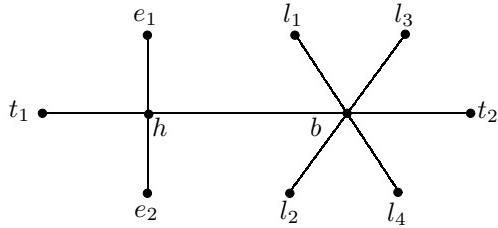
A *graph*  $G$  is an ordered 2-tuple  $(V, E)$  with  $V \neq \emptyset$  and  $E \subset V \times V$ , where  $V$  and  $E$  are finite sets and respectively called the vertex set, the edge set of  $G$ , denoted by  $V(G)$  or  $E(G)$ , and a graph  $G$  is said to be embeddable into a topological space  $\mathcal{T}$  if there is a  $1 - 1$  continuous mapping  $\phi : G \rightarrow \mathcal{T}$  with  $\phi(p) \neq \phi(q)$  if  $p, q \notin V(G)$ . Particularly, if  $\mathcal{T} = \mathbb{R}^3$  such a topological

graph is called *spacial graph* such as those shown in Fig.10 for cube  $C_4 \times C_4$ ,



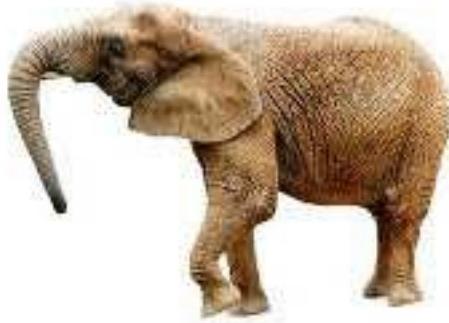
**Fig.10**

and a labeling on a graph  $G$  is a mapping  $L : V(G) \cup E(G) \rightarrow \mathcal{L}$  with a labeling set  $\mathcal{L}$ . For example,  $\mathcal{L} = \{v_i, u_i, e_j, 1 \leq i \leq 4, 1 \leq j \leq 12\}$  in Fig.10. Notice that the inherent structure of an elephant finding by these blind men is a labeled tree shown in Fig.11,



**Fig.11**

where,  $\{t_1\}$  =tusk,  $\{e_1, e_2\}$  =ears,  $\{h\}$  =head,  $\{b\}$  =belly,  $\{l_1, l_2, l_3, l_4\}$  =legs and  $\{t_2\}$  =tail. Then, *how can one rebuilt the geometric space of an elephant from the labeled tree in space  $\mathbb{R}^3$ ?* First, one can blows up all edges, i.e.,  $e \rightarrow$  a cylinder for  $\forall e \in E(G^L)$  and then, homeomorphically transforms these cylinders as parts of an elephant. After these transformations, a 3-dimensional elephant is built again in  $\mathbb{R}^3$  such as those shown in Fig.12.



**Fig.12**

All of these discussions implies that labeled graph should be a mathematical element for understanding things ([20]), not only a labeling game because of

$$\text{Labeled Graphs in } \mathbb{R}^n \Leftrightarrow \text{Inherent Structure of Things.}$$

But *what are labels on labeled graphs, is it just different symbols?* And *are such labeled graphs a mechanism for the reality of things, or only a labeling game?* In fact, labeled graphs are researched mainly on symbols, not mathematical elements. If one puts off this assumption, i.e., labeling a graph by elements in mathematical systems, *what will happens? Are these resultants important for understanding things in the world?* The answer is certainly yes ([6], [7]) because this step will enable one to pullback more characters of things, particularly the metrics in physics, characterize things precisely and then holds on the reality of things.

### 3.2 G-Solutions on Equations

Let  $\mathcal{F} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  be a  $\mathbb{C}^k$ ,  $1 \leq k \leq \infty$  mapping with  $\mathcal{F}(\bar{x}_0, \bar{y}_0) = \bar{0}$  for  $\bar{x}_0 \in \mathbb{R}^n$ ,  $\bar{y}_0 \in \mathbb{R}^m$  and a non-singular  $m \times m$  matrix  $(\partial \mathcal{F}^j / \partial y^i(\bar{x}_0, \bar{y}_0))$ . Then the implicit mapping theorem concludes that there exist opened neighborhoods  $V \subset \mathbb{R}^n$  of  $\bar{x}_0$ ,  $W \subset \mathbb{R}^m$  of  $\bar{y}_0$  and a  $\mathbb{C}^k$  mapping  $\phi : V \rightarrow W$  such that  $T(\bar{x}, \phi(\bar{x})) = \bar{0}$ , i.e., equation (1.2) is always solvable.

Let  $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m$  be  $m$  mappings holding with conditions of the implicit mapping theorem and let  $S_{\mathcal{F}_i} \subset \mathbb{R}^n$  be a manifold such that  $\mathcal{F}_i : S_{\mathcal{F}_i} \rightarrow 0$  for integers  $1 \leq i \leq m$ . Consider the equations

$$\begin{cases} \mathcal{F}_1(x_1, x_2, \dots, x_n) = 0 \\ \mathcal{F}_2(x_1, x_2, \dots, x_n) = 0 \\ \dots \dots \dots \dots \\ \mathcal{F}_m(x_1, x_2, \dots, x_n) = 0 \end{cases} \quad (3.1)$$

in Euclidean space  $\mathbb{R}^n$ ,  $n \geq 1$ . Geometrically, the system (3.1) is non-solvable or not dependent on  $\bigcap_{i=1}^m S_{\mathcal{F}_i} = \emptyset$  or  $\neq \emptyset$ .

Now, *is the non-solvable case meaningless for understanding the reality of things?* Certainly not because the non-solvable case of (3.1) only concludes the intersection  $\bigcap_{i=1}^m S_{\mathcal{F}_i} = \emptyset$ , the behavior of the solvable and non-solvable cases should be both characterized by the union  $\bigcup_{i=1}^m S_{\mathcal{F}_i}$  such as those shown in (2.2) for the elephant.

For example, if things  $T_1, T_2, T_3, T_4$  and  $T'_1, T'_2, T'_3, T'_4$  are respectively characterized by systems of equations following

$$(LES_4^N) \quad \begin{cases} x + y = 1 \\ x + y = -1 \\ x - y = -1 \\ x - y = 1 \end{cases} \quad (LES_4^S) \quad \begin{cases} x = y \\ x + y = 2 \\ x = 1 \\ y = 1 \end{cases}$$

it is clear that  $(LES_4^N)$  is non-solvable because  $x + y = -1$  is contradicition to  $x + y = 1$ , and so that for equations  $x - y = -1$  and  $x - y = 1$ , i.e., there are no solutions  $x_0, y_0$  hold with this system. But  $(LES_4^S)$  is solvable with  $x = 1$  and  $y = 1$ . *Can we conclude that things  $T'_1, T'_2, T'_3, T'_4$  are  $x = 1, y = 1$  and  $T_1, T_2, T_3, T_4$  are nothing?* Certainly not because

$(x, y) = (1, 1)$  is the intersection of straight line behavior of things  $T'_1, T'_2, T'_3, T'_4$  and there are no intersection of  $T_1, T_2, T_3, T_4$  in plane  $\mathbb{R}^2$ . However, they are indeed exist in  $\mathbb{R}^2$  such as those shown in Fig.13.

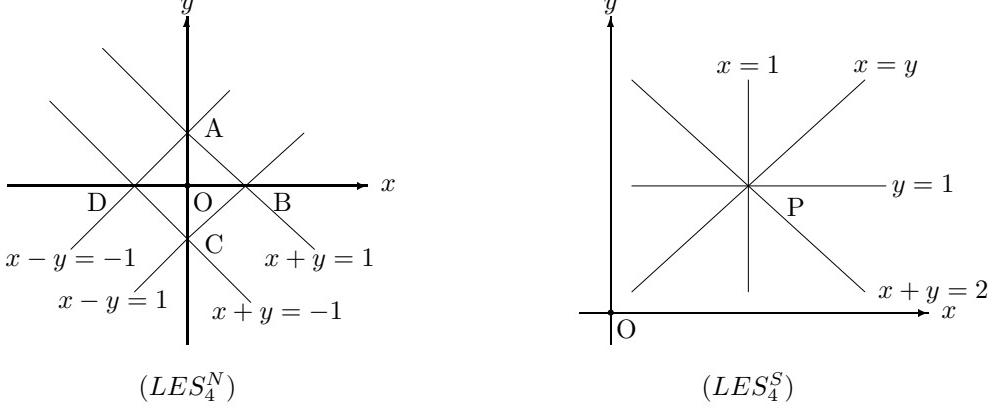


Fig.13

Let  $L_{a,b,c} = \{(x, y) | ax + by = c, ab \neq 0\}$  be points in  $\mathbb{R}^2$ . We are easily know the straight line behaviors of  $T_1, T_2, T_3, T_4$  and  $T'_1, T'_2, T'_3, T'_4$  are nothings else but the unions  $L_{1,-1,0} \cup L_{1,1,2} \cup L_{1,0,1} \cup L_{0,1,1}$  and  $L_{1,1,1} \cup L_{1,1,-1} \cup L_{1,-1,-1} \cup L_{1,-1,1}$ , respectively.

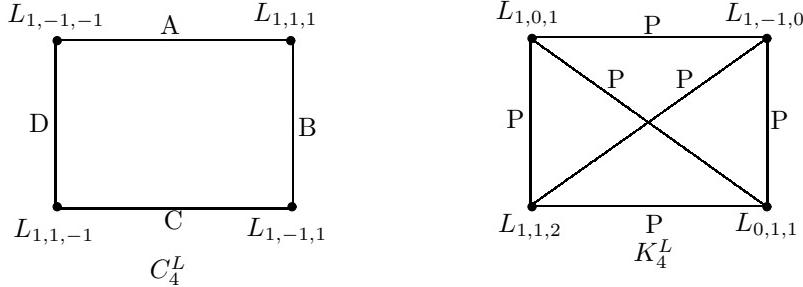


Fig.14

**Definition 3.1** A  $G$ -solution of system (3.1) is a labeling graph  $G^L$  defined by

$$V(G) = \{S_{\mathcal{F}_i}, 1 \leq i \leq n\};$$

$$E(G) = \{(S_{\mathcal{F}_i}, S_{\mathcal{F}_j}) \text{ if } S_{\mathcal{F}_i} \cap S_{\mathcal{F}_j} \neq \emptyset \text{ for integers } 1 \leq i, j \leq n\} \text{ with a labeling}$$

$$L : S_{\mathcal{F}_i} \rightarrow S_{\mathcal{F}_i}, \quad (S_{\mathcal{F}_i}, S_{\mathcal{F}_j}) \rightarrow S_{\mathcal{F}_i} \cap S_{\mathcal{F}_j}.$$

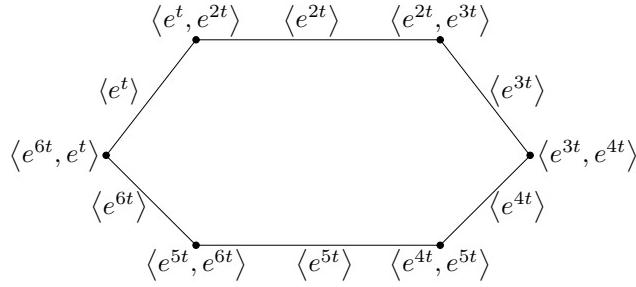
For Example, the  $G$ -solutions of  $(LES_4^N)$  and  $(LES_4^S)$  are respectively labeling graphs  $C_4^L$  and  $K_4^L$  shown in Fig.14. Generally, we know the following result.

**Theorem 3.2** A system (3.1) of equations is  $G$ -solvable if  $\mathcal{F}_i \in \mathbb{C}^1$  and  $\mathcal{F}_i|_{(x_1^0, x_2^0, \dots, x_n^0)} = 0$  but  $\frac{\partial \mathcal{F}_i}{\partial x_i}|_{(x_1^0, x_2^0, \dots, x_n^0)} \neq 0$  for any integer  $i$ ,  $1 \leq i \leq n$ .

More results on combinatorics of non-solvable algebraic, ordinary or partial differential equations can be found in references [9]-[14]. For example, let  $(LDES_m^1)$  be a system of linear homogeneous differential equations

$$\left\{ \begin{array}{ll} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{array} \right.$$

where  $\ddot{x} = \frac{d^2x}{dt^2}$  and  $\dot{x} = \frac{dx}{dt}$ . Clearly, this system is non-solvable with solution bases  $\{e^t, e^{2t}\}$ ,  $\{e^{2t}, e^{3t}\}$ ,  $\{e^{3t}, e^{4t}\}$ ,  $\{e^{4t}, e^{5t}\}$ ,  $\{e^{5t}, e^{6t}\}$ ,  $\{e^{6t}, e^t\}$  respectively on equations (1) – (6) and its  $G$ -solution is shown in Fig.15,



**Fig.15**

where  $\langle \Delta \rangle$  denotes the linear space generalized by elements in  $\Delta$ .

### 3.3 Mathematics Over Graph

Let  $(\mathcal{A}; \circ_1, \circ_2, \dots, \circ_k)$  be an algebraic system, i.e.,  $a \circ_i b \in \mathcal{A}$  for  $\forall a, b \in \mathcal{A}$ ,  $1 \leq i \leq k$  and let  $\vec{G}$  be an oriented graph embedded in space  $T$ . Denoted by  $\vec{G}_{\mathcal{A}}^L$  all of those labeled graphs  $\vec{G}^L$  with labeling  $L : E(\vec{G}) \rightarrow \mathcal{A}$  constraint with ruler:

**R1 :** For  $\forall \vec{G}^{L_1}, \vec{G}^{L_2} \in \vec{G}_{\mathcal{A}}^L$ , define  $\vec{G}^{L_1} \circ_i \vec{G}^{L_2} = \vec{G}^{L_1 \circ_i L_2}$ , where  $L_1 \circ_i L_2 : e \rightarrow L_1(e) \circ_i L_2(e)$  for  $\forall e \in E(\vec{G})$  and integers  $1 \leq i \leq k$ .

For example, such a ruler on graph  $\vec{C}_4$  is shown in Fig.16, where  $\mathbf{a}_3 = \mathbf{a}_1 \circ_i \mathbf{a}_2$ ,  $\mathbf{b}_3 = \mathbf{b}_1 \circ_i \mathbf{b}_2$ ,  $\mathbf{c}_3 = \mathbf{c}_1 \circ_i \mathbf{c}_2$ ,  $\mathbf{d}_3 = \mathbf{d}_1 \circ_i \mathbf{d}_2$ .

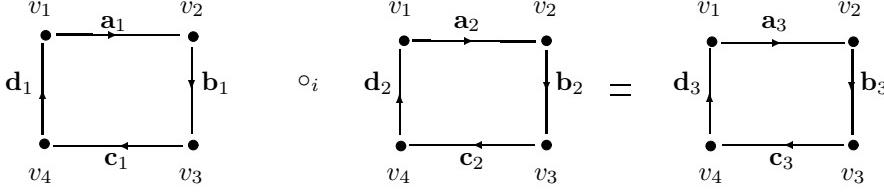


Fig.16

Then,  $\vec{G}^{L_1} \circ_i \vec{G}^{L_2} = \vec{G}^{L_1 \circ_i L_2} \in \vec{G}_{\mathcal{A}}^L$  by the ruler **R1**, and generally,

$$\vec{G}^{L_1} \circ_{i_1} \vec{G}^{L_2} \circ_{i_2} \dots \circ_{i_s} \vec{G}^{L_{s+1}} \in \vec{G}_{\mathcal{A}}^L$$

for integers  $1 \leq i_1, i_2, \dots, i_s \leq k$ , i.e.,  $\vec{G}_{\mathcal{A}}^L$  is also an algebraic system, and it is commutative on an operation  $\circ_i$  if  $(\mathcal{A}; \circ_1, \circ_2, \dots, \circ_k)$  is commutative on an operation  $\circ_i$  for an integer  $i$ ,  $1 \leq i \leq k$ . Particularly, if  $k = 1$ ,  $\vec{G}_{\mathcal{A}}^L$  is a group if  $(\mathcal{A}; \circ_1)$  is a group. Thus, we extend  $(\mathcal{A}; \circ_1, \circ_2, \dots, \circ_k)$  and obtain an algebraic system over graph  $\vec{G}$  underlying a geometrical structure in space  $T$ .

Notice that such an extension  $\vec{G}_{\mathcal{A}}^L$  is only a pure extension of algebra over  $\vec{G}$  without combining the nature of things, i.e., the conservation of matter which states that the amount of the conserved quantity at a point or within a volume can only change by the amount of the quantity which flows in or out of that volume. Thus, understanding the reality of things motivates the extension of mathematical systems  $(\mathcal{A}; \circ_1, \circ_2, \dots, \circ_k)$  over graph  $\vec{G}$  constrained also on the laws of conservation

**R2**:  $\sum_l \mathbf{F}(v)_l^- = \sum_s \mathbf{F}(v)_s^+$ , where  $\mathbf{F}(v)_l^-$ ,  $l \geq 1$  and  $\mathbf{F}(v)_s^+$ ,  $s \geq 1$  denote respectively the output and input amounts at vertex  $v \in E(\vec{G})$ .

This notion brings about a new mathematical element finally, i.e., action flows, which combines well the continuous mathematics with that of the discrete.

**Definition 3.3([19])** An action flow  $(\vec{G}; L, A)$  is an oriented embedded graph  $\vec{G}$  in a topological space  $\mathcal{S}$  associated with a mapping  $L : (v, u) \rightarrow L(v, u)$ , 2 end-operators  $A_{vu}^+ : L(v, u) \rightarrow L^{A_{vu}^+}(v, u)$  and  $A_{uv}^+ : L(u, v) \rightarrow L^{A_{uv}^+}(u, v)$  on a Banach space  $\mathcal{B}$

$$u \xrightarrow{A_{uv}^+} L(u, v) \xrightarrow{A_{vu}^+} v$$

Fig.17

with  $L(v, u) = -L(u, v)$  and  $A_{vu}^+(-L(v, u)) = -L^{A_{vu}^+}(v, u)$  for  $\forall (v, u) \in E(\vec{G})$  holding with conservation laws

$$\sum_{u \in N_G(v)} L^{A_{vu}^+}(v, u) = \mathbf{c}_v \text{ for } \forall v \in V(\vec{G})$$

such as those shown for vertex  $v$  in Fig.18 following

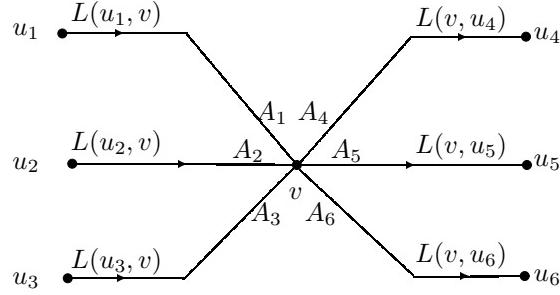


Fig.18

with a conservation law

$$-L^{A_1}(v, u_1) - L^{A_2}(v, u_2) - L^{A_4}(v, u_3) + L^{A_4}(v, u_4) + L^{A_5}(v, u_5) + L^{A_6}(v, u_6) = \mathbf{c}_v,$$

where  $\mathbf{c}_v$  is the surplus flow on vertex  $v$ , and usually, let  $\mathbf{c}_v = \mathbf{0}$ .

Indeed, action flow is an element both with the character of continuous and discrete mathematics. For example, the conservation laws on an action flow over dipole shown in Fig.19

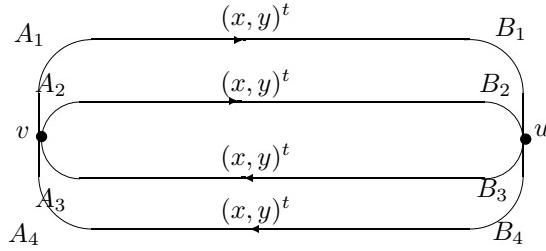


Fig.19

are partial differential equations

$$\begin{cases} a_1 \frac{\partial^2 x}{\partial t^2} + b_1 \frac{\partial^2 y}{\partial t^2} - a_3 \frac{\partial x}{\partial t} + (a_2 - a_4)x + (b_2 - b_3 - b_4)y = 0 \\ c_2 \frac{\partial^2 x}{\partial t^2} + d_2 \frac{\partial^2 y}{\partial t^2} - d_4 \frac{\partial y}{\partial t} + (c_1 - c_3 - c_4)x + (d_1 - d_3)y = 0 \end{cases},$$

where,  $A_1 = (a_1 \partial^2 / \partial t^2, b_1 \partial^2 / \partial t^2)$ ,  $A_2 = (a_2, b_2)$ ,  $A_3 = (a_3 \partial / \partial t, b_3)$ ,  $A_4 = (a_4, b_4)$ ,  $B_1 = (c_1, d_1)$ ,  $B_2 = (c_2 \partial^2 / \partial t^2, d_2 \partial^2 / \partial t^2)$ ,  $B_3 = (c_3, d_3)$ ,  $B_4 = (c_4, d_4 \partial / \partial t)$ .

Certainly, not all mathematical systems can be extended over a graph  $\vec{G}$  constraint with the laws of conservation at  $v \in V(\vec{G})$  unless  $\vec{G}$  with special structure but such an extension of linear space  $\mathcal{A}$  can be always done.

**Theorem 3.4([20])** Let  $(\mathcal{A}; +, \cdot)$  be a linear space,  $\vec{G}$  an embedded graph in space  $T$  and  $A_{vu}^+ = A_{uv}^+ = \mathbf{1}_{\mathcal{A}}$  for  $\forall(v, u) \in E(\vec{G})$ . Then,  $(\vec{G}_{\mathcal{A}}^L; +, \cdot)$  is also a linear space under rulers

**R1** and **R2** with dimension  $\dim \mathcal{A}^{\beta(\vec{G})}$  if  $\dim \mathcal{V} < \infty$ , where  $\beta(\vec{G}) = |E(\vec{G})| - |V(\vec{G})| + 1$ , or infinite.

An action flow  $(\vec{G}; L, \mathbf{1}_{\mathcal{A}})$ , i.e.,  $A_{vu}^+ = A_{uv}^+ = \mathbf{1}_{\mathcal{A}}$  for  $\forall(v, u) \in E(\vec{G})$  is usually called  $\vec{G}$ -flows, denoted by  $\vec{G}^L$  and the linear space  $(\vec{G}_{\mathcal{A}}^L; +, \cdot)$  extended over  $\vec{G}$  by  $\vec{G}^{\mathcal{A}}$  for simplicity.

## §4. Banach $\vec{G}$ -Flow Spaces with Multiverses

### 4.1 Banach $\vec{G}$ -Flow Space

A Banach or Hilbert space is respectively a linear space  $\mathcal{A}$  over a field  $\mathbb{R}$  or  $\mathbb{C}$  equipped with a complete norm  $\|\cdot\|$  or inner product  $\langle \cdot, \cdot \rangle$ , i.e., for every Cauchy sequence  $\{x_n\}$  in  $\mathcal{A}$ , there exists an element  $x$  in  $\mathcal{A}$  such that

$$\lim_{n \rightarrow \infty} \|x_n - x\|_{\mathcal{A}} = 0 \quad \text{or} \quad \lim_{n \rightarrow \infty} \langle x_n - x, x_n - x \rangle_{\mathcal{A}} = 0,$$

which can be extended over graph  $\vec{G}$  by introducing the norm of a  $\vec{G}$ -flow  $\vec{G}^L$  following

$$\left\| \vec{G}^L \right\| = \sum_{(v, u) \in E(\vec{G})} \|L(v, u)\|,$$

where  $\|L(v, u)\|$  is the norm of  $L(v, u)$  in  $\mathcal{A}$ .

**Theorem 4.1([15])** *For any graph  $\vec{G}$ ,  $\vec{G}^{\mathcal{A}}$  is a Banach space, and furthermore, if  $\mathcal{A}$  is a Hilbert space,  $\vec{G}^{\mathcal{A}}$  is a Hilbert space too.*

We can also consider operators action on the Banach or Hilbert  $\vec{G}$ -flow space  $\vec{G}^{\mathcal{A}}$ . Particularly, an operator  $\mathbf{T} : \vec{G}^{\mathcal{A}} \rightarrow \vec{G}^{\mathcal{A}}$  is *linear* if

$$\mathbf{T}(\lambda \vec{G}^{L_1} + \mu \vec{G}^{L_2}) = \lambda \mathbf{T}(\vec{G}^{L_1}) + \mu \mathbf{T}(\vec{G}^{L_2})$$

for  $\forall \vec{G}^{L_1}, \vec{G}^{L_2} \in \vec{G}^{\mathcal{A}}$ ,  $\lambda, \mu \in \mathbb{F}$ , which enables one to generalize the representation theorem of Fréchet and Riesz on linear continuous functionals of Hilbert space to Hilbert  $\vec{G}$ -flow space  $\vec{G}^{\mathcal{A}}$  following.

**Theorem 4.2([15])** *Let  $\mathbf{T} : \vec{G}^{\mathcal{A}} \rightarrow \mathbb{C}$  be a linear continuous functional. Then there is a unique  $\vec{G}^{\hat{L}} \in \vec{G}^{\mathcal{A}}$  such that  $\mathbf{T}(\vec{G}^L) = \langle \vec{G}^L, \vec{G}^{\hat{L}} \rangle$  for  $\forall \vec{G}^L \in \vec{G}^{\mathcal{A}}$ .*

Notice that linear continuous functionals exist everywhere in mathematics, particularly, the differential and integral operators. For example, let  $\mathcal{A}$  be a Hilbert space consisting of measurable functions  $f(x_1, x_2, \dots, x_n)$  on a set

$$\Delta = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n | a_i \leq x_i \leq b_i, 1 \leq i \leq n\},$$

which is a functional space  $L^2[\Delta]$  with inner product

$$\langle f(\mathbf{x}), g(\mathbf{x}) \rangle = \int_{\Delta} \overline{f(\mathbf{x})} g(\mathbf{x}) d\mathbf{x} \quad \text{for } f(\mathbf{x}), g(\mathbf{x}) \in L^2[\Delta],$$

where  $\mathbf{x} = (x_1, x_2, \dots, x_n)$ . By Theorem 4.1,  $\mathcal{A}$  can be extended to Hilbert  $\vec{G}$ -flow space  $\vec{G}^{\mathcal{A}}$ , and the *differential* or *integral operators*

$$D = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad \int_{\Delta}$$

on  $\mathcal{A}$  are extended to  $\vec{G}^{\mathcal{A}}$  respectively by  $D\vec{G}^L = \vec{G}^{DL(v,u)}$  and

$$\int_{\Delta} \vec{G}^L = \int_{\Delta} K(\mathbf{x}, \mathbf{y}) \vec{G}^{L[\mathbf{y}]} d\mathbf{y} = \vec{G}^{\int_{\Delta} K(\mathbf{x}, \mathbf{y}) L(v, u)[\mathbf{y}] d\mathbf{y}}$$

for  $\forall (v, u) \in E(\vec{G})$ , where  $a_i, \frac{\partial a_i}{\partial x_j} \in \mathbb{C}^0(\Delta)$  for integers  $1 \leq i, j \leq n$  and  $K(\mathbf{x}, \mathbf{y}) : \Delta \times \Delta \rightarrow \mathbb{C} \in L^2(\Delta \times \Delta, \mathbb{C})$  with

$$\int_{\Delta \times \Delta} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} < \infty.$$

**Theorem 4.3([15])** *The differential or integral operator  $D : \vec{G}^{\mathcal{A}} \rightarrow \vec{G}^{\mathcal{A}}$ ,  $\int_{\Delta} : \vec{G}^{\mathcal{A}} \rightarrow \vec{G}^{\mathcal{A}}$  both are linear operators on  $\vec{G}^{\mathcal{A}}$ .*

For example, let  $f(t) = t$ ,  $g(t) = e^t$ ,  $K(t, \tau) = t^2 + \tau^2$  for  $\Delta = [0, 1]$  and let  $\vec{G}^L$  be the  $\vec{G}$ -flow shown on the left in Fig.20. Then we get the  $\vec{G}$ -flows on the right in Fig.20 by the differential or integral operator action on.

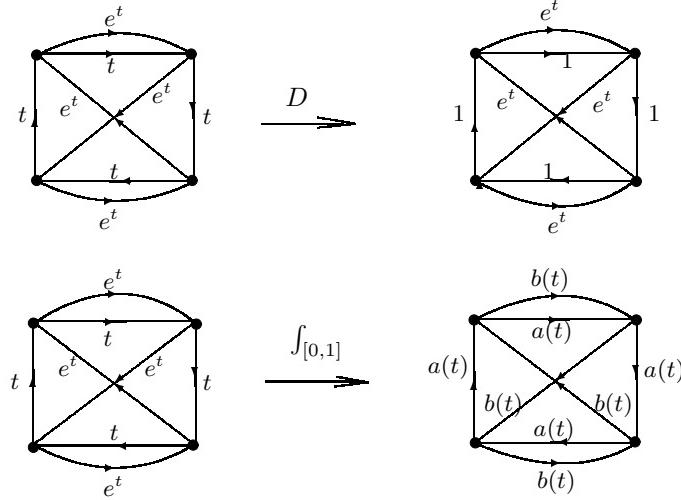


Fig.20

where  $a(t) = \frac{t^2}{2} + \frac{1}{4}$  and  $b(t) = (e - 1)t^2 + e - 2$ .

## 4.2 Multiverses on Equations

Notice that solving Schrödinger equation (1.3) with initial data only get one state of a particle  $P$  but the particle is in superposition, which brought the H.Everett multiverse on superposition and the quark model of Sakata, or Gell-Mann and Ne'eman on particles machinery. However, Theorems 4.1 – 4.3 enables one to get multiverses constraint with linear equations (3.1) in  $\vec{G}^{\mathcal{A}}$ .

For example, we can consider the Cauchy problem

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^n \frac{\partial^2 X}{\partial x_i^2}$$

with initial values  $X|_{t=t_0}$  in  $\vec{G}^{\mathbb{R}^n \times \mathbb{R}}$ , i.e., Hilbert space  $\mathbb{R}^n \times \mathbb{R}$  over graph  $\vec{G}$ , and get multiverse solutions of heat equation following.

**Theorem 4.4([15])** *For  $\forall \vec{G}^{L'} \in \vec{G}^{\mathbb{R}^n \times \mathbb{R}}$  and a non-zero constant  $c$  in  $\mathbb{R}$ , the Cauchy problems on differential equation*

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^n \frac{\partial^2 X}{\partial x_i^2}$$

*with initial value  $X|_{t=t_0} = \vec{G}^{L'} \in \vec{G}^{\mathbb{R}^n \times \mathbb{R}}$  is solvable in  $\vec{G}^{\mathbb{R}^n \times \mathbb{R}}$  if  $L'(v, u)$  is continuous and bounded in  $\mathbb{R}^n$  for  $\forall (v, u) \in E(\vec{G})$ .*

And then, the H.Everett's multiverse on the Schrödinger equation (1.3) is nothing else but a 2-branch tree

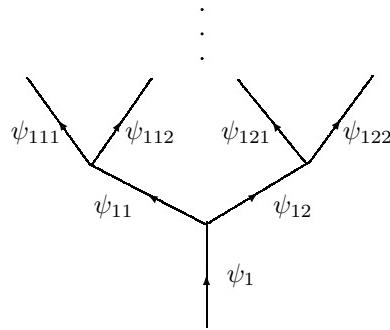


Fig.21

with equalities  $\psi_1 = \psi_{11} + \psi_{12}$ ,  $\psi_{11} = \psi_{111} + \psi_{112}$ ,  $\psi_{12} = \psi_{121} + \psi_{122}$ ,  $\dots$  ([16], [17]).

If the equations (3.1) is not linear, we can not immediately apply Theorems 4.1 – 4.3 to get multiverse over any graphs  $\vec{G}$ . However, if the graph  $\vec{G}$  is prescribed with special structures, for instance the circuit decomposable, we can also solve the Cauchy problem on an equation in Hilbert  $\vec{G}$ -flow space  $\vec{G}^{\mathcal{A}}$  if it is solvable in  $\mathcal{A}$  and obtain a general conclusion following, which enable us to interpret also the superposition of particles ([17]), biological diversity and establish multiverse model of spacetime in Einstein's gravitation.

**Theorem 4.5([15])** If the graph  $\vec{G}$  is strong-connected with circuit decomposition  $\vec{G} = \bigcup_{i=1}^l \vec{C}_i$  such that  $L(v, u) = L_i(\mathbf{x})$  for  $\forall(v, u) \in E(\vec{C}_i)$ ,  $1 \leq i \leq l$  and the Cauchy problem

$$\begin{cases} \mathcal{F}_i(\mathbf{x}, u, u_{x_1}, \dots, u_{x_n}, u_{x_1 x_2}, \dots) = 0 \\ u|_{x_0} = L_i(\mathbf{x}) \end{cases}$$

is solvable in a Hilbert space  $\mathcal{A}$  on domain  $\Delta \subset \mathbb{R}^n$  for integers  $1 \leq i \leq l$ , then the Cauchy problem

$$\begin{cases} \mathcal{F}_i(\mathbf{x}, X, X_{x_1}, \dots, X_{x_n}, X_{x_1 x_2}, \dots) = 0 \\ X|_{x_0} = \vec{G}^L \end{cases}$$

such that  $L(v, u) = L_i(\mathbf{x})$  for  $\forall(v, u) \in E(\vec{C}_i)$  is solvable for  $\vec{G}^L \in \vec{G}^{\mathcal{A}}$ .

Theorem 4.5 enables one to explore the multiverse, particularly, the solutions of Einstein's gravitational equations

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \lambda g^{\mu\nu} = -8\pi G T^{\mu\nu},$$

where  $R^{\mu\nu} = R_{\alpha}^{\mu\alpha\nu} = g_{\alpha\beta} R^{\alpha\mu\beta\nu}$ ,  $R = g_{\mu\nu} R^{\mu\nu}$  are the respective Ricci tensor, Ricci scalar curvature,  $G = 6.673 \times 10^{-8} \text{ cm}^3/\text{gs}^2$ ,  $\kappa = 8\pi G/c^4 = 2.08 \times 10^{-48} \text{ cm}^{-1} \cdot \text{g}^{-1} \cdot \text{s}^2$ . In fact, Einstein's general relativity is established on  $\mathbb{R}^4$ . However, if the dimension  $n$  of the universe > 4, how can we characterize the structure of spacetime for the universe? In fact, if the dimension of the universe > 4, all observations are nothing else but a projection of the true faces on our six organs because the dimension of human beings is 3 hold with

$$\mathbb{R}^n = \bigcup_{i=1}^m \mathbb{R}_i^4 \quad \text{and} \quad \left| \bigcap_{i=1}^m \mathbb{R}_i^4 \right| = 1$$

such as those shown in Fig.22 for a projection of 3-dimensional objects on Euclidean plane  $\mathbb{R}^2$ .

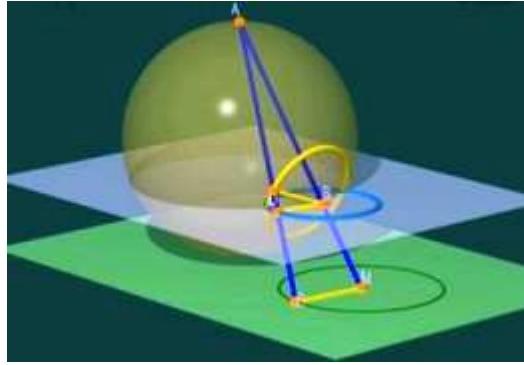


Fig.22

In this case, we can characterize the spacetime with a complete graph  $K_m^L$  labeled by  $\mathbb{R}^4$  ([7]-[8]). For example, if  $m = 4$  there are 4 Einstein's gravitational equations respectively on

$v \in V(K_4^L)$ . We can solve them one by one by applying the spherically symmetric solution in  $\mathbb{R}^4$  and construct  $K_4^L$  shown in Fig.23,

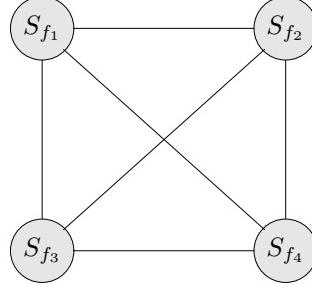


Fig.23

where, each  $S_{\mathcal{F}_i}$  is the geometrical space of the Schwarzschild spacetime

$$ds^2 = f(t) \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{1}{1 - \frac{r_s}{r}} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

for integers  $1 \leq i \leq 4$ .

Notice that  $m = 4$  is only an assumption. We do not know the exact value of  $m$  at present. Similarly, by Theorem 4.5 we can also get a conclusion on multiverse of the Einstein's gravitational equations, even in  $\mathbb{R}^4$ . Certainly, we do not know also which is the real spacetime of the universe.

**Theorem 4.6([15], [19])** *There are infinite many  $\vec{G}$ -flow solutions on Einstein's gravitational equations*

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -8\pi G T^{\mu\nu}$$

in  $\vec{G}^C$ , particularly on those graphs with circuit-decomposition  $\vec{G} = \bigcup_{i=1}^m \vec{C}_i$  labeled with Schwarzschild spacetime on their edges.

For example, let  $\vec{G} = \vec{C}_4$ . We are easily find  $\vec{C}_4$ -flow solution of Einstein's gravitational equations such as those shown in Fig.24.

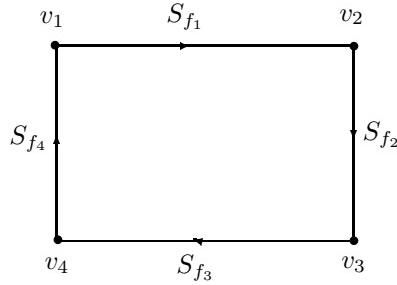
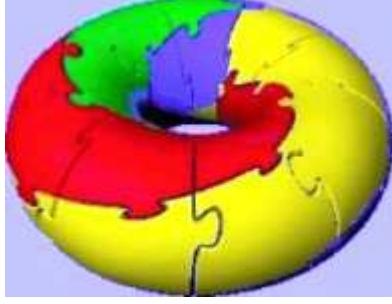


Fig.24

Then, the spacetime of the universe is nothing else but a curved ring in space such as those shown in Fig.25.

**Fig.25**

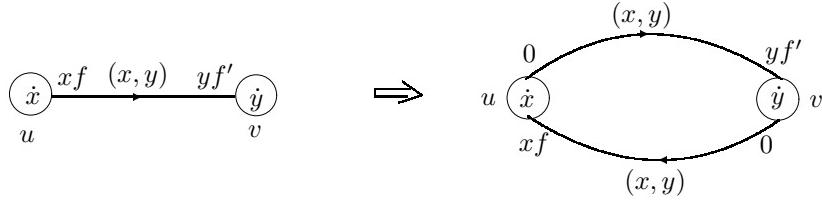
Generally, if  $\vec{G}$  is the union of  $m$  orientated circuits  $\vec{C}_i$ ,  $1 \leq i \leq m$ , Theorem 4.6 implies the spacetime of Einstein's gravitational equations is a multiverse consisting of  $m$  curved rings over graph  $\vec{G}$  in space.

Notice that a graph  $\vec{G}$  is circuit decomposable if and only if it is an Eulerian graph. Thus, Theorems 4.1–4.5 can be also applied to biology with global stability of food webs on  $n$  species following.

**Theorem 4.7([21])** *A food web  $\vec{G}^L$  with initial value  $\vec{G}^{L_0}$  is globally stable or asymptotically stable if and only if there is an Eulerian multi-decomposition*

$$(\vec{G} \cup \overleftarrow{G})^{\hat{L}} = \bigoplus_{i=1}^s \vec{H}_i^L$$

with solvable stable or asymptotically stable conservative equations on Eulerian subgraphs  $\vec{H}_i^L$  for integers  $1 \leq i \leq s$ , where  $(\vec{G} \cup \overleftarrow{G})^{\hat{L}}$  is the bi-digraph of  $\vec{G}$  defined by  $\vec{G} \cup \overleftarrow{G}$  with a labeling  $\hat{L} : V(\vec{G} \cup \overleftarrow{G}) \rightarrow L(V(\vec{G}))$ ,  $\hat{L} : E(\vec{G} \cup \overleftarrow{G}) \rightarrow L(E(\vec{G} \cup \overleftarrow{G}))$  by  $\hat{L} : (u, v) \rightarrow \{0, (x, y), yf'\}$ ,  $(v, u) \rightarrow \{xf, (x, y), 0\}$  if  $L : (u, v) \rightarrow \{xf, (x, y), yf'\}$  for  $\forall(u, v) \in E(\vec{G})$ , such as those shown in Fig.26,

**Fig.26**

and a multi-decomposition  $\bigoplus_{i=1}^s \vec{H}_i^L$  of  $(\vec{G} \cup \overleftarrow{G})^{\hat{L}}$  is defined by

$$(\vec{G} \cup \overleftarrow{G})^{\hat{L}} = \bigcup_{i=1}^s \vec{H}_i$$

with  $\vec{H}_i \neq \vec{H}_j$ ,  $\vec{H}_i \cap \vec{H}_j = \emptyset$  or  $\neq \emptyset$  for integers  $1 \leq i \neq j \leq s$ .

**Theorem 4.8([21])** A food web  $\vec{G}^L$  with initial value  $\vec{G}^{L_0}$  is globally asymptotically stable if there is an Eulerian multi-decomposition

$$\left(\vec{G} \bigcup \vec{G}\right)^{\hat{L}} = \bigoplus_{k=1}^s \vec{H}_k^L$$

with solvable conservative equations such that  $\operatorname{Re}\lambda_i < 0$  for characteristic roots  $\lambda_i$  of  $A_v$  in the linearization

$$A_v X_v = 0_{h_v \times h_v}$$

of conservative equations at an equilibrium point  $\vec{H}_k^{L_0}$  in  $\vec{H}_k^L$  for integers  $1 \leq i \leq h_v$  and  $v \in V(\vec{H}_k^L)$ , where  $V(\vec{H}_k^L) = \{v_1, v_2, \dots, v_{h_v}\}$ ,

$$A_v = \begin{pmatrix} a_{11}^v & a_{12}^v & \cdots & a_{1h_v}^v \\ a_{21}^v & a_{22}^v & \cdots & a_{2h_v}^v \\ \cdots & \cdots & \cdots & \cdots \\ a_{h1}^v & a_{h2}^v & \cdots & a_{hh_v}^v \end{pmatrix}$$

a constant matrix and  $X_k = (x_{v_1}, x_{v_2}, \dots, x_{v_{h_v}})^T$  for integers  $1 \leq k \leq l$ .

## §5. Conclusion

Answer the question which is better to the reality of things on the continuous or discrete mathematics is not easy because our world appears both with the continuous and discrete characters. However, contradictions exist everywhere which are all artificial, not the nature of things. Thus, holding on the reality of things which are in fact being a complex system motivates one to turn mathematical contradictory systems to compatible systems, i.e., giving up the notion that contradiction is meaningless and establish an envelope theory on mathematics, which needs the combination of the continuous mathematics with that of discrete, i.e., mathematical combinatorics because a non-mathematics in classical is in fact a mathematics over a graph  $\vec{G}$  (See [13] for details).

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## The First Zagreb Index, Vertex-Connectivity, Minimum Degree And Independent Number in Graphs

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**Abstract:** Let  $G$  be a simple, undirected and connected graph. Defined by  $M_1(G)$  and  $RMTI(G)$  the first Zagreb index and the reciprocal Schultz molecular topological index of  $G$ , respectively. In this paper, we determined the graphs with maximal  $M_1$  among all graphs having prescribed vertex-connectivity and minimum degree, vertex-connectivity and bipartition, vertex-connectivity and vertex-independent number, respectively. As applications, all maximal elements with respect to  $RMTI$  are also determined among the above mentioned graph families, respectively.

**Key Words:** Zagreb index, reciprocal molecular topological index, vertex connectivity, bipartite graph, independent number.

**AMS(2010):** 05C12.

### §1. Introduction

Let  $G$  be a simple graph with vertex set  $V(G)$  and edge set  $E(G)$ . The degree of a vertex  $v$  is the number of edges incident to  $v$  and denoted by  $d(v)$ . One of the most important topological indices is the well-known Zagreb indices introduced in [8, 10], the first and second Zagreb indices  $M_1$  and  $M_2$  of  $G$ , respectively, are defined as follows:

$$M_1(G) = \sum_{v \in V(G)} d(v)^2, M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

They reflect the extent of branching of the underlying molecular structure [8, 10, 20]. Their main properties were recently summarized in [1, 4, 6, 7, 9, 11, 12, 13, 15, 16, 23, 24, 25, 26].

Let  $G$  be a connected graph with  $n$  vertices. The distance matrix  $\mathbf{D} = (D_{ij})_{n \times n}$  of  $G$  is an  $n \times n$  matrix such that  $D_{ij}$  is the distance between vertices  $i$  and  $j$  in  $G$  [18]. The reciprocal distance matrix  $\mathbf{R}$ , also called the Harary matrix (see [14, 18]), is defined as an  $n \times n$  matrix  $R = (R_{ij})$  such that  $R_{ij} = \frac{1}{D_{ij}}$  if  $i \neq j$  and 0 otherwise. Let  $R_i = \sum_{j=1}^n R_{ij}$ . Then the

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<sup>1</sup>Received October 12, 2016, Accepted May 14, 2017.

reciprocal molecular topological index  $RMTI$  [20] of  $G$  is defined as

$$RMTI(G) = \sum_{i=1}^n R_i^2 + \sum_{i=1}^n d_i R_i.$$

Some formulations of reciprocal and constant-interval reciprocal Schultz-type topological indices, included  $RMTI$ , have been discussed in [20], and they were illustrated by the QSPR, which studied on physical constants of alkanes and cycloalkanes.

Recently, Zhou and Trinajstić [27] reported some properties of the reciprocal molecular topological index  $RMTI$ . They also derived the upper bounds for  $RMTI$  in terms of the number of vertices and the number of edges for various classes of graphs under some restricted conditions.

In this paper, we determined, respectively, the graphs with maximal value of  $M_1$  among all graphs having prescribed graph invariants, such as, vertex-connectivity and minimum degree, vertex-connectivity and vertex-independent number. As applications, all maximum elements with respect to  $RMTI(G)$  are also determined among the above mentioned graph families, respectively.

## §2. Preliminaries

Denoted by  $\delta(G)$  the minimum degree of  $G$ , and by  $Diam(G)$  the diameter of a graph  $G$ , i.e., the maximum cardinality among all distance of any one pair of vertices in  $G$ . Let  $K_n$  be the complete graph with  $n$  vertices. Suppose that  $G_1$  and  $G_2$  are graphs with  $V(G_1) \cap V(G_2) = \emptyset$ . Denoted by  $G_1 \cup G_2$  the new graph with vertex set  $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$  and edge set  $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$ . The join of  $G_1$  and  $G_2$ , denoted by  $G_1 \vee G_2$ , is the new graph with vertex set  $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$  and edge set  $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{xy | x \in V(G_1) \text{ and } y \in V(G_2)\}$ .

For  $S, S' \subseteq V(G)$ , the induced subgraph of  $S$ , denoted by  $G[S]$ , is the graph whose vertex set is  $S$  and edge set is composed of those edges with both ends in  $S$ . The induced subgraph of  $S$  and  $S'$ , denoted by  $G[S, S']$ , is the graph whose vertex set is  $S_1 \cup S_2$  and edge set is composed of those edges with one end in  $S$  and another end in  $S'$ .

The bipartite graph is the graph whose vertices can be divided into two disjoint sets  $U$  and  $V$ , such that every edge connects a vertex in  $U$  to one in  $V$ . Vertex sets  $U$  and  $V$  usually called the parts of the graph. A vertex cut of a connected graph  $G$  is a set of vertices whose removal renders  $G$  disconnected. The vertex-connectivity  $\kappa(G)$  is the size of a minimal vertex cut. An independent set of  $G$  is a set of vertices in a graph  $G$ , no two of which are adjacent. A maximum independent set is an independent set of largest possible size for a given graph  $G$ . This size is called the independence number of  $G$ , and denoted by  $\alpha(G)$ .

For other notations and terminology not defined here, see [5].

By the definition of the first Zagreb index, the lemma follows immediately.

**Lemma 2.1** *Let  $G$  be a simple graph with  $u, v \in V(G)$  and  $uv \notin E(G)$ . Then*

$$M_1(G + uv) > M_1(G).$$

**Lemma 2.2([27])** *Let  $G$  be a connected simple graph with  $n$  vertices and  $m$  edges. Then*

$$RMTI(G) \leq \frac{3}{2}M_1(G) + (n - 1)m,$$

*with equality holds if and if  $Diam(G) \leq 2$ .*

### §3. Graphs with Given Connectivity and Minimum Degree

Let  $n, k$  and  $\delta$  be integers such that  $n \geq \delta \geq k \geq 1$ . Denoted by  $\mathcal{G}(n, k, \delta)$  the set of  $n$ -vertex connected graphs with vertex-connectivity  $k$  and minimum degree  $\delta$ , where  $1 \leq k \leq \delta$  and  $2 \leq \delta \leq n$ .

**Theorem 3.1** *If  $G \in \mathcal{G}(n, k, \delta)$  with  $k \leq \delta \leq n - 1$ . Then*

$$M_1(G) \leq n(n - 1)^2 + (n - k)(k + \delta - 2n + 3)(k + \delta + 1)$$

*with equality holds if and only if  $G = K_k \vee (K_{\delta-k+1} \cup K_{n-\delta-1})$ .*

*Proof* If  $n = k + 1$ , then  $k = \delta = n - 1$ , i.e.,  $\mathcal{G}(n, k, \delta) = \{K_{k+1}\}$ . Suppose that  $n \geq k + 2$ . Let  $G_{max}$  be graph in  $\mathcal{G}(n, k, \delta)$  with maximal  $M_1$ - value in  $\mathcal{G}(n, k, \delta)$ , that is,  $M_1(G) \leq M_1(G_{max})$  for all  $G \in \mathcal{G}(n, k, \delta)$ . Denoted by  $S \subset V(G_{max})$  the vertex cut and  $|S| = k$ . We will prove the three claims as follows.

**Claim 1.**  $G_{max} - S$  contain exactly two components.

**Proof of Claim 1:** Suppose by contrary that  $G_{max} - S$  contain at least three components. Denoted two components of  $G_{max} - S$  by  $C_1$  and  $C_2$ . There exist vertices  $u \in V(C_1), v \in V(C_2)$  such that  $G_{max} + uv \in \mathcal{G}(n, k, \delta)$ . By Lemma 2.1,  $M_1(G_{max} + uv) > M_1(G_{max})$ , which contradicts the choice of  $G_{max}$ . This completes the proof of Claim 1.

Therefore, we assume that  $G_{max} - S$  contain exactly two connected components, denoted by  $C_1$  and  $C_2$ . Denoted by  $|V(C_1)| = n_1, |V(C_2)| = n_2$ . Since  $\delta \leq d(u) \leq n_1 - 1 + k$  and  $\delta \leq d(v) \leq n_2 - 1 + k$  for  $u \in V(C_1), v \in V(C_2)$ , we have  $n_1, n_2 \geq \delta - k + 1$ .

**Claim 2.**  $G_{max}[S \cup V(C_1)]$  and  $G_{max}[S \cup V(C_2)]$  are cliques.

**Proof of Claim 2:** Without loss of generality, suppose by contrary that  $G_{max}[S \cup V(C_1)]$  is not a clique. There are two cases as follows:

**Case 1.** There exists nonadjacent vertices  $u, v \in S \cup V(C_1)$  such that  $G_{max} + uv \in \mathcal{G}(n, k, \delta)$ , then by Lemma 2.1,  $M_1(G_{max} + uv) > M_1(G_{max})$ , which contradicts the choice of  $G_{max}$ .

**Case 2.** Otherwise, adding a new edge to  $G_{max}$  will increase the minimum degree of  $G$ . From Eqn.(1) in the proof of Claim 3, we have

$$M_1(G_{max}) < M_1(K_s \vee (K_{n_1} \cup K_{n_2})) \leq M_1(K_k \vee (K_{\delta-k+1} \cup K_{n-\delta-1})),$$

which contradicts the choice of  $G_{max}$  since  $K_k \vee (K_{\delta-k+1} \cup K_{n-\delta-1}) \in \mathcal{G}(n, k, \delta)$ .

This complete the proof of Claim 2.

From Claim 2, we suppose that  $G_{max} = K_k \vee (K_{n_1} \cup K_{n_2})$ , where  $n_1, n_2 \geq 1$  and  $n_1 + n_2 = n - k$ .

**Claim 3.**  $n_1 = \delta - k + 1$  or  $n_2 = \delta - k + 1$ .

**Proof of Claim 3:** Consider the graph  $K_k \vee (K_{n_1} \cup K_{n_2})$ . Suppose by contrary that  $n_1 \geq n_2 > \delta - k + 1$ , by direct calculation, we have

$$\begin{aligned} M_1(K_k \vee (K_{n_1+1} \cup K_{n_2-1})) &> M_1(K_k \vee (K_{n_1} \cup K_{n_2})) \\ &= k(n-1)^2 + n_1(k+n_1-1)^2 + (n-k-n_1)(n-n_1-1)^2, \end{aligned} \quad (1)$$

which implies that  $M_1(K_k \vee (K_{\delta-k+1} \cup K_{n-\delta-1})) > M_1(K_k \vee (K_{n_1} \cup K_{n_2}))$  if  $n_1, n_2 > \min\{\delta - k + 1, n - \delta - 1\}$ . This complete the proof of Claim 3.

By combine above claims, we have  $G_{max} = K_k \vee (K_{\delta-k+1} \cup K_{n-\delta-1})$ . Then the result holds.  $\square$

**Corollary 3.1** Let  $G \in \mathcal{G}(n, k, \delta)$  with  $m$  edges and  $k \leq \delta \leq n - 1$ . Then

$$RMTI(G) \leq \frac{3}{2}n(n-1)^2 + \frac{3}{2}n(n-k)(k+\delta-2n+3)(k+\delta+1) + (n-1)m$$

with equality if and only if  $k = n - 1$  and  $G = K_{k+1}$ .

#### §4. Bipartite Graphs with Given Connectivity

Let  $\mathcal{B}(n, k)$  be the set of bipartite graphs with  $n$  vertices and  $\kappa(G) = k$ , and  $B_{n,x}$  the graph obtained from  $K_{x,n-x-1}$  by adding a new vertex  $v$  to  $k$  vertices of degree  $x$  of  $K_{x,n-x-1}$ .

**Theorem 4.1** Let  $G \in \mathcal{B}(n, k)$  with  $1 \leq k \leq n - 1$ . Then

$$M_1(G) \leq \max\{f(a), f(b)\},$$

with equality if and only if  $G \in \{B_{n,a}, B_{n,b}\}$ , where

$$\begin{aligned} f(x) &= nx(n-x) - x(2n+2k+1) + k(k-1), \\ a &= \left\lfloor \frac{(n-1)^2 - 2(k+1)}{2n} \right\rfloor \text{ and} \\ b &= \left\lceil \frac{(n-1)^2 - 2(k+1)}{2n} \right\rceil. \end{aligned}$$

*Proof* If  $k = 1$ , then  $\mathcal{B}(n, k) = \{K_{1,n-1}\}$ . Suppose that  $1 < k \leq \frac{n}{2}$ , let  $G_{max}$  be the graph with the maximal  $M_1$ -value in  $\mathcal{B}(n, k)$ , and  $S$  a  $k$ -vertex cut of  $G_{max}$ . Let  $A, B$  be vertex parts of  $V(G_{max})$  such that  $A \cup B = V(G_{max})$ . Denoted by  $S_A = S \cap A, S_B = S \cap B$ . We will

prove the four claims as follows.

**Claim 1.**  $G_{max}[S]$  and  $G_{max}[C \cup S]$  are complete bipartite graphs, where  $C$  is one of components in  $G_{max} - S$ .

**Proof of Claim 1:** Suppose by contrary that  $G_{max}[S]$  or  $G_{max}[C \cup S]$  is not a complete bipartite graph. There exist vertices  $u, v \in V(G_{max})$  and  $uv \notin E(G_{max})$  such that  $G_{max} + uv \in \mathcal{B}(n, k)$ . By Lemma 2.1, we have  $M_1(G_{max} + uv) > M_1(G_{max})$ , which contradicts the choice of  $G_{max}$ . This complete the proof of Claim 1.

**Claim 2.** If  $S_A \neq \emptyset$  and  $S_B \neq \emptyset$ , then  $G_{max} - S$  have exactly two components.

**Proof of Claim 2:** Suppose by contrary that  $G_{max} - S$  contain at least three components. Let  $C_1$  and  $C_2$  be two components of  $G_{max} - S$ . Then there exist vertices  $u \in V(C_1) \cap A$ ,  $v \in V(C_2) \cap B$  such that  $G_{max} + uv \in \mathcal{B}(n, k)$  and  $S$  is also a  $k$ -vertex cut of  $G_{max} + uv$ . By Lemma 2.1,  $M_1(G_{max} + uv) > M_1(G_{max})$ , which contradicts the choice of  $G_{max}$ . Thus Claim 2 holds.

**Claim 3.**  $S_A = \emptyset$  or  $S_B = \emptyset$ .

**Proof of Claim 3:** Suppose by contrary that  $S_A \neq \emptyset$  and  $S_B \neq \emptyset$ . From Claim 2,  $G_{max} - S$  contain exactly two components, denoted by  $C_1, C_2$ . Let  $u \in V(C_1) \cap A$  and  $v \in V(C_2) \cap B$ . Without loss of generality, we assume that  $a = d(u) \geq d(v) = b > 0$  and  $|N_{C_2}(v)| = c > 0$ . Taking transformations on  $G_{max}$  as follows.

(1) Let  $G_1 = G_{max} - \{wv : w \in N_{C_2}(v)\} + \{wu : w \in N_{C_2}(v)\}$ . Then

$$M_1(G_1) - M_1(G_{max}) = (b+c)^2 + (a-c)^2 - (b^2 + a^2) = 2c(b-a+c) > 0,$$

(2) Consider the graph  $G_1$ . Using the definitions from  $G_{max}$ . Let  $|S_B| = s$ , and choose arbitrary vertices  $v_1, v_2, \dots, v_{k-s} \in B - S$ . Let  $G_2$  be the graph obtained from  $G_1$  by adding more edges between  $A - \{v\}$  and  $B$  as possible, and then adding edges  $v_1v, v_2v, \dots, v_{k-s}v$ . It is obviously that  $N_{G_2}(v)$  is the vertex cut of  $G_2$  and  $|N_{G_2}(v)| = k$ , i.e.,  $G_2 \in \mathcal{B}(n, k)$ . From 1) of Claim 3 and Lemma 2.1, we have

$$M_1(G_2) > M_1(G_{max}),$$

which contradicts the choice of  $G_{max}$ . Thus Claim 3 holds.

From Claim 3, without loss of generality, let  $S \subset A$  be the  $k$ -vertex cut of  $G_{max}$ .

**Claim 4.**  $G_{max} - S$  contains a isolated vertex.

**Proof of Claim 4:** From Claim 1, suppose by contrary that the components of  $G_{max} - S$ , denoted by  $C_1, C_2$ , are complete bipartite graphs. Let  $V(C_1) = A_1 \cup B_1$  and  $V(C_2) = A_2 \cup B_2$ , where  $A_i, B_i$  are vertex parts of  $C_i$ , i.e.,  $A_i \subset A, B_i \subset B, i = 1, 2$ .

Without loss of generality, suppose that  $S \subset A$ . Let  $u \in B_1$ . Let  $G^*$  be the graph obtained from  $G_{max}$  by deleting the edges connecting  $u$  and vertices in  $A_1$ , and adding more edges between  $A - S$  and  $B - \{u\}$  as possible, i.e.,  $G^* = G_{max} - \{xu : x \in A_1\} + \{xy : x \in A - S, y \in B - \{u\}\}$ .

$B - \{u\}, xy \notin E(G_{max})\}$ . Then  $S$  is also a  $k$ -vertex cut of  $G_2$ , and  $G_2 \in \mathcal{B}(n, k)$ . Similar to Claim 3(1), we have

$$M_1(G^*) - M_1(G_{max}) > 0,$$

which contradicts the choice of  $G_{max}$ . Thus  $G_{max} = B_{n,x}$ .

By calculation, we have

$$f(x) = M_1(B_{n,x}) = nx(n-x) - x(2n+2k+1) + k(k-1)$$

and  $x = \frac{(n-1)^2 - 2(k+1)}{2n}$ , and can obtain its maximal value by differentiating  $f(x)$  on  $x$ . Since  $k \leq \frac{(n-1)^2 - 2(k+1)}{2n} \leq n-2$ , let

$$a = \left\lfloor \frac{(n-1)^2 - 2(k+1)}{2n} \right\rfloor \text{ and } b = \left\lceil \frac{(n-1)^2 - 2(k+1)}{2n} \right\rceil.$$

Then by Claim 4, we have  $G_{max} \in \{B_{n,a}, B_{n,b}\}$ . This completes the proof.  $\square$

**Corollary 4.1** *Let  $G \in \mathcal{B}(n, k)$  with  $m$  edges and  $1 \leq k \leq n-1$ . Then*

$$RMTI(G) \leq \frac{3}{2}T + (n-1)m,$$

where

$$\begin{aligned} T &= \max\{f(a), f(b)\}, \quad f(x) = nx(n-x) - x(2n+2k+1) + k(k-1), \\ a &= \left\lfloor \frac{(n-1)^2 - 2(k+1)}{2n} \right\rfloor \text{ and} \\ b &= \left\lceil \frac{(n-1)^2 - 2(k+1)}{2n} \right\rceil. \end{aligned}$$

## §5. Graphs with Given Connectivity and Independent Number

Let  $\mathcal{D}(n, k, r)$  be the set of  $n$ -vertex graphs with  $\kappa(G) = k$  and  $\alpha(G) = r$ .

**Theorem 5.1** *Let  $G \in \mathcal{D}(n, k, r)$  with  $r \geq 1$  and  $1 \leq k \leq n-1$ . Then*

$$M_1(G) \leq (r-1)(n-r)^2 + (n-r)(n-2)^2 + k^2 + k(2n-3),$$

with equalities hold if and only if  $G = K_k \vee (K_1 \cup (K_{n-k-r} \vee (r-1)K_1))$ .

*Proof* If  $r = 2$ , then  $M_1(G) \leq M_1(K_k \vee (K_1 \cup K_{n-k-1}))$  from [11]. We assume that  $2 \leq r \leq n-1$ , let  $G_{max}$  be the graph with the maximal  $M_1$ -value in  $\mathcal{D}(n, k, r)$ . Denoted by  $S$  the  $k$ -vertex cut of  $G_{max}$ , and by  $D$  the maximum independent set of  $G_{max}$ . We will prove three claims as follows.

**Claim 1.**  $G_{max}[C] = K_a \vee (c-a)K_1$ ,  $G_{max}[S] = K_b \vee (k-b)K_1$  and  $G_{max}[S \cup V(C)] = K_{a+b} \cup (k+c-a-b)K_1$ , where  $C$  is one of components of  $G_{max}-S$ ,  $|V(C)| = c$ ,  $|V(C)-D| = a$  and  $|S-D| = b$ .

**Proof of Claim 1:** By Lemma 2.1 and the definition of  $G_{max}$ , it clear that  $G_{max} = G_{max}[S] \vee G_{max}[V(G)-S]$ . Now suppose by contrary that  $G_{max}[C] \neq K_a \vee (c-a)K_1$ . There exist  $u, v \in V(C)-D$  and  $w \in V(C) \cap D$ , such that  $uv \notin E(G_{max})$  or  $uw \notin E(G_{max})$ . It is clear that  $G_{max}+uv, G_{max}+uw \in \mathcal{D}(n, k, r)$ . By Lemma 2.1, we have  $M_1(G_{max}+uv) > M_1(G_{max})$  or  $M_1(G_{max}+uw) > M_1(G_{max})$ , which contradicts the choice of  $G_{max}$ . Similarly to  $S$ , we have  $G_{max}[S] = K_b \vee (k-b)K_1$ . Thus Claim 1 holds.

**Claim 2.**  $G_{max}-S$  contain exactly two components.

**Proof of Claim 2:** Suppose by contrary that  $G_{max}-S$  contain at least three components. Let  $C_1$  and  $C_2$  be two of components, and  $u \in V(C_1)-D, v \in V(C_2)-D$ . Then  $G_{max}+uv \in \mathcal{D}(n, k, r)$ . By Lemma 2.1,  $M_1(G_{max}+uv) > M_1(G_{max})$ , which contradicts the choice of  $G_{max}$ . Thus Claim 2 holds.

By Claim 2,  $G_{max}-S = C_1 \cup C_2$ , where  $C_1$  and  $C_2$  are components of  $G_{max}-S$ .

**Claim 3.** If  $V(C_1) \geq V(C_2)$ , then  $|V(C_2)| = 1$ .

**Proof of Claim 3:** Suppose by contrary that  $|V(C_2)| \geq 2$ . If  $V(C_2)-D = \emptyset$ , then  $|V(C_2)| = 1$  since  $C_2$  is a connected components. Suppose that  $V(C_2)-D \neq \emptyset$ , then  $V(C_2) \cap D \neq \emptyset$ . Otherwise,  $V(C_2) \cap D = \emptyset$ , choose  $u \in V(C_2)$ , and  $D \cup \{u\}$  is a independent set, which contradicts the definition of  $D$ .

Using the definitions from  $G_{max}$  and constructing a new graph  $G^*$  as follows. Let  $v \in V(C_2) \cap D$ . Then

$$G^* = G_{max} - \{xv : x \in V(C_2)-D\} + \{xy : x \in V(C_2)-\{v\}, y \in V(C_1)\},$$

it is clear that  $S$  and  $D$  are also minimal vertex cut and maximal independent set of  $G^*$ , respectively. Thus  $G^* \in \mathcal{D}(n, k, r)$ .

Let  $u \in V(G_{max})-S-\{v\}$  and  $w \in V(C_1)-D$ . Then  $d_{G^*}(u) > d_{G_{max}}(u)$  and

$$M_1(G^*) - M_1(G_{max}) > d_{G^*}(w)^2 + d_{G^*}(v)^2 - d_{G_{max}}(w)^2 - d_{G_{max}}(v)^2 > 0,$$

which contradicts the choice of  $G_{max}$ . Thus Claim 3 holds.

By combine above claims, we have

$$G_{max} \in \{G' : G' = (K_{n-r} \vee (r-1)K_1) \cup \{v\} \cup \{u_i v : u_i \in S, i = 1, 2, \dots, k\}\},$$

where  $v$  is a isolated vertex of  $G_{max}-S$ . Let  $|S \cap D| = a$ . Then

$$\begin{aligned} M_1(G') &= a(n-r+1)^2 + (r-a-1)(n-r)^2 + (k-a)(n-1)^3 \\ &\quad + (n-r-k+a)(n-2)^2 + k^2 \\ &= (r-1)(n-r)^2 + (n-r)(n-2)^2 + k^2 + k(2n-3) - a(2r-4), \end{aligned}$$

and the point  $a = 0$  attains the maximal value of  $M_1(G)$ . Therefore,  $M_1(G_{max}) = (r-1)(n-r)^2 + (n-r)(n-2)^2 + k^2 + k(2n-3)$  and  $G_{max} = K_k \vee (K_1 \cup (K_{n-k-1} \vee (r-1)K_1))$ . This complete the proof.  $\square$

**Corollary 5.1** *Let  $G \in \mathcal{D}(n, k, r)$  with  $m$  edges,  $r \geq 1$  and  $1 \leq k \leq n-1$ . Then*

$$RMTI(G) \leq \frac{3}{2} [(r-1)(n-r)^2 + (n-r)(n-2)^2 + k^2 + k(2n-3)] + (n-1)m.$$

### Acknowledgement

This work was supported by NSF of Guangdong Province of China (2015A030313574, 2014A030-310119, 2014B020004027), the Outstanding Young Teacher Training Program in Guangdong Universities (YQ2015155), the National Social Science Fund in China (15BTJ024) and the Scientific research innovation team Project of Huizhou University (HZUXl201523).

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## D-Conformal Curvature Tensor in Generalized $(\kappa, \mu)$ -Space Forms

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**Abstract:** The object of the first two sections is to give brief history of generalized  $(\kappa, \mu)$  space forms and some basic results related to such manifold. In the last section we have derived few results regarding  $D$ -conformal curvature tensor in generalized  $(\kappa, \mu)$  space-forms.

**Key Words:** Generalized  $(\kappa, \mu)$ -space form,  $D$ -conformal curvature tensor,  $\eta$ -Einstein manifold.

**AMS(2010):** 53B25, 53C15, 53C25, 53D15.

### §1. Introduction

In [1], Carriazo jointly with P. Alegre and D.E. Blair defined a generalized Sasakian space form as an almost contact metric manifold  $(M, \phi, \xi, \eta, g)$  whose curvature tensor  $R$  is given by

$$\begin{aligned} R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ &\quad + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\} \end{aligned} \tag{1.1}$$

for any vector fields  $X, Y, Z$  on  $M$ .

In particular a Sasakian manifold  $M(\phi, \xi, \eta, g)$  is said to be a Sasakian space form if all the  $\phi$ -sectional curvatures  $K(X \wedge \phi X)$  are equal to a constant  $c$ , where  $K(X \wedge \phi X)$  denotes the sectional curvature of the section spanned by the unit vector field  $X$ , orthogonal to  $\xi$  and  $\phi X$ . Later on many scientists R. Al-Ghefari, F. R. Alsomy [2],[5], M. H. Shahid have studied the CR-submanifolds of generalized Sasakian space forms. After them Ricci curvature of contact CR-submanifolds of such space were studied in [6].

In [2] authors studied contact metric and generalized Sasakian-space forms. In [7] and [8] authors studied locally  $\phi$ -symmetric and  $\eta$ -recurrent Ricci tensor and also studied the projective curvature tensor respectively. Generalized Sasakian space form with few properties like conformally flat, locally symmetric were studied by Kim [9].

In recent paper [10], the authors (jointly with M. M. Tripathi) defined a generalized  $(\kappa, \mu)$ -space form as an almost contact metric manifold  $(M^{2n+1}, \phi, \xi, \eta, g)$  whose curvature tensor is

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<sup>1</sup>Received July 12, 2016, Accepted May 16, 2017.

given as

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6, \quad (1.2)$$

where  $f_1, f_2, f_3, f_4, f_5, f_6$  are differentiable functions on  $M$ , and  $R_1, R_2, R_3, R_4, R_5, R_6$  are tensors defined as follows:

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \quad (1.3)$$

$$R_2(X, Y)Z = g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z, \quad (1.4)$$

$$R_3(X, Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi, \quad (1.5)$$

$$R_4(X, Y)Z = g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y, \quad (1.6)$$

$$R_5(X, Y)Z = g(hY, Z)hX - g(hX, Z)hY + g(\phi hX, Z)\phi hY - g(\phi hY, Z)\phi hX, \quad (1.7)$$

$$R_6(X, Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX, Z)\eta(Y)\xi - g(hY, Z)\eta(X)\xi, \quad (1.8),$$

where  $2h = \mathcal{L}_\xi\phi$  and  $L$  is the usual Lie derivative. Usually, this manifold is denoted by  $M(f_1, f_2, f_3, f_4, f_5, f_6)$ . If  $f_4 = f_5 = f_6 = 0$  then the manifold is the usual Sasakian space form.

Again,  $(\kappa, \mu)$ -space forms are natural examples of generalized  $(\kappa, \mu)$  space forms for constant functions ([10])

$$f_1 = \frac{c+3}{4}, \quad f_2 = \frac{c-1}{4}, \quad f_3 = \frac{c+3}{4}, \quad f_4 = 1, \quad f_5 = \frac{1}{2}, \quad f_6 = 1 - \mu. \quad (1.9)$$

In this paper we have established few conditions related to  $D$ -conformal curvature tensor.

## §2. Preliminaries

An almost contact metric manifold is a  $(2n+1)$ -dimensional manifold endowed with an almost contact structure  $(\phi, \xi, \eta)$  consisting of a tensor field  $\phi$  of type  $(1, 1)$ , a structure vector field  $\xi$  and 1-form  $\eta$  satisfying:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad (2.1)$$

for any vector field  $X, Y \in \tilde{M}$  and a Riemannian metric  $g$  defined as

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (2.2)$$

From above equation we can easily derive

$$g(X, \xi) = \eta(X). \quad (2.3)$$

The metric tensor satisfies the following properties:

$$g(\phi X, Y) = -g(X, \phi Y), \quad (2.4)$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y). \quad (2.5)$$

In a  $(2n + 1)$  dimensional generalized  $(\kappa, \mu)$ -space form we obtain from (1.2)

$$\begin{aligned} R(X, Y)\xi &= f_1\{g(Y, \xi)X - g(X, \xi)Y\} \\ &\quad + f_2\{g(X, \phi\xi)\phi Y - g(Y, \phi\xi)\phi X + 2g(X, \phi Y)\phi\xi\} \\ &\quad + f_3\{\eta(X)\eta(\xi)Y - \eta(Y)\eta(\xi)X + g(X, \xi)\eta(Y)\xi - g(Y, \xi)\eta(X)\xi\} \\ &\quad + f_4\{g(Y, \xi)hX - g(X, \xi)hY + g(hY, \xi)X - g(hX, \xi)Y\} \\ &\quad + f_5g(hY, \xi)hX - g(hX, \xi)hY + g(\phi hX, \xi)\phi hY - g(\phi hY, \xi)\phi hX \\ &\quad + f_6\eta(X)\eta(\xi)hY - \eta(Y)\eta(\xi)hX + g(hX, \xi)\eta(Y)\xi - g(hY, \xi)\eta(X)\xi \end{aligned} \tag{2.6}$$

After some brief calculations we obtain from [4]

$$R(X, Y)\xi = (f_1 - f_3)\{\eta(X)Y - \eta(Y)X\} + (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\}. \tag{2.7}$$

Now putting  $X = \xi$ ,  $Y = X$ ,  $Z = Y$  we get

$$R(\xi, X)Y = (f_1 - f_3)\{g(X, Y)\xi - \eta(Y)X\} + (f_4 - f_6)\{g(hX, Y) - \eta(Y)hX\}. \tag{2.8}$$

Again putting  $Y = \xi$  in (2.7) we get

$$R(\xi, X)\xi = (f_1 - f_3)\{\eta(X)\xi - X\} - (f_4 - f_6)\{hX\}. \tag{2.9}$$

Applying  $\eta$  on both side of the equation (1.2) we calculate

$$\begin{aligned} \eta(R(X, Y)Z) &= (f_1 - f_3)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} \\ &\quad + (f_4 - f_6)\{g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)\}. \end{aligned} \tag{2.10}$$

Putting  $Z = \xi$  we can easily write

$$\eta(R(X, Y)\xi) = 0. \tag{2.11}$$

Applying  $\eta$  on both side of equation (2.7) we can get the following equations

$$\begin{aligned} \eta(R(\xi, X)Y) &= (f_1 - f_3)\{g(X, Y) - \eta(Y)\eta(X)\} \\ &\quad + (f_4 - f_6)g(hX, Y), \end{aligned} \tag{2.12}$$

$$\begin{aligned} S(X, Y) &= \{2nf_1 + 3f_2 - f_3\}g(X, Y)\{(2n - 1)f_4 - f_6\}g(hX, Y) \\ &\quad - \{3f_2 + (2n - 1)f_3\}\eta(X)\eta(Y). \end{aligned} \tag{2.13}$$

From (2.13) we obtain

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \tag{2.14}$$

$$r = 2n\{(2n + 1)f_1 + 3f_2 - 2f_3\}, \tag{2.15}$$

$$\begin{aligned} S(\phi X, \phi Y) &= S(X, Y) - 2n(f_1 - f_3)\eta(X)\eta(Y) \\ &\quad - \{(2n-1)f_4 - f_6\}g(hX, Y), \end{aligned} \tag{2.16}$$

$$\begin{aligned} QX &= \{2nf_1 + 3f_2 - f_3\}X + \{(2n-1)f_4 - f_6\}hX - \{3f_2 \\ &\quad + (2n-1)f_3\}\eta(X)\xi, \end{aligned} \tag{2.17}$$

$$Q\xi = 2n(f_1 - f_3)\xi. \tag{2.18}$$

From [12],  $D$ -conformal curvature tensor on a Riemannian manifold  $(M^{2n+1}, g)$  is defined as

$$\begin{aligned} B(X, Y)Z &= R(X, Y)Z + \frac{1}{2(n-1)}\{S(X, Z)Y - S(Y, Z)X + g(X, Z)QY \\ &\quad - g(Y, Z)QX\} - S(X, Z)\eta(Y)\xi + S(Y, Z)\eta(X)\xi - \eta(X)\eta(Z)QY \\ &\quad - \eta(Y)\eta(Z)QX\} - \frac{k-2}{2(n-1)}\{g(X, Z)Y - g(Y, Z)X\} \\ &\quad + \frac{k}{2(n-1)}\{g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \eta(X)\eta(Z)Y \\ &\quad - \eta(Y)\eta(Z)X\}, \end{aligned} \tag{2.19}$$

where  $k = \frac{r+4n}{2n-1}$ ,  $R$  is the curvature tensor,  $S$  is the Ricci tensor and  $r$  is the scalar curvature.

Now we give the definition of  $D$ -conformally flat generalized  $(\kappa, \mu)$  space form following.

**Definition 2.1** A  $(2n+1)$ -dimensional generalized  $(\kappa, \mu)$  space form  $M(f_1, f_2, f_3, f_4, f_5, f_6)$  is said to be  $D$ -conformally flat if

$$B(X, Y)Z = 0. \tag{2.20}$$

We give the definition of  $\xi - D$ -conformally flat generalized  $(\kappa, \mu)$  space form following.

**Definition 2.2** A  $(2n+1)$ -dimensional generalized  $(\kappa, \mu)$  space form  $M(f_1, f_2, f_3, f_4, f_5, f_6)$  is said to be  $\xi - D$ -conformally flat if

$$B(X, Y)\xi = 0. \tag{2.21}$$

Also we mention the following definition.

**Definition 2.3** A  $(2n+1)$ -dimensional generalized  $(\kappa, \mu)$  space form  $M(f_1, f_2, f_3, f_4, f_5, f_6)$  is said to be  $\phi - D$ -conformally flat if

$$g(B(\phi X, \phi Y)\phi Z, \phi W) = 0. \tag{2.22}$$

### §3. Main Results

From Definition 2.1 we can draw the following theorem.

**Theorem 3.1** If a  $(2n + 1)$ -dimensional generalized  $(\kappa, \mu)$  space form  $M(f_1, f_2, f_3, f_4, f_5, f_6)$  is  $D$ -conformally flat then  $f_3 = f_1 + 1$  and  $f_4 = f_6$ .

*Proof* Let us consider a  $(2n + 1)$ -dimensional generalized  $(\kappa, \mu)$  space form which satisfy the condition  $B(X, Y)Z = 0$  then from (2.19) we obtain on using Definition 2.1 and taking inner product with  $W$  we obtain

$$\begin{aligned} 0 &= R(X, Y, Z, W) + \frac{1}{2(n-1)} \{ S(X, Z)g(Y, W) - S(Y, Z)g(X, W) + g(X, Z)g(QY, W) \\ &\quad - g(Y, Z)g(QX, W) \} - S(X, Z)\eta(Y)\eta(W) + S(Y, Z)\eta(X)\eta(W) - \eta(X)\eta(Z)g(QY, W) \\ &\quad - \eta(Y)\eta(Z)g(QX, W) \} - \frac{k-2}{2(n-1)} \{ g(X, Z)g(Y, W) - g(Y, Z)g(X, W) \} \\ &\quad + \frac{k}{2(n-1)} \{ g(X, Z)\eta(Y)\eta(W) - g(Y, Z)\eta(X)\eta(W) + \eta(X)\eta(Z)g(Y, W) \\ &\quad - \eta(Y)\eta(Z)g(X, W) \} \end{aligned} \tag{3.1}$$

because of  $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ .

Now setting  $W = \xi$  in (3.1) and using (2.1) and (2.2), we have

$$\begin{aligned} 0 &= \eta(R(X, Y)Z) + \frac{1}{2(n-1)} \{ S(Y, \xi)g(X, Z) - S(X, \xi)g(Y, Z) - S(Y, \xi)\eta(X)\eta(Z) \\ &\quad + S(X, \xi)\eta(Y)\eta(Z) + 2\{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\} \}. \end{aligned} \tag{3.2}$$

On using (2.10) and (2.14) we get on brief calculation

$$\begin{aligned} 0 &= \frac{f_3 - f_1 - 1}{n-1} \{ g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \} + (f_4 - f_6) \{ g(hY, Z)\eta(X) \\ &\quad - g(hX, Z)\eta(Y) \}. \end{aligned} \tag{3.3}$$

Since L.H.S. is equal to zero and

$$\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \{g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)\} \neq 0$$

we must have  $f_3 - f_1 - 1 = 0$  and  $f_4 - f_6 = 0$ .

Hence

$$f_3 = f_1 + 1 = 0, \quad f_4 = f_6. \tag{3.4}$$

Therefore the above equation proves our theorem.  $\square$

Now on basis of the definition (2.2) we give our next theorem.

**Corollary 3.1** If a  $(2n + 1)$ -dimensional generalized  $(\kappa, \mu)$  space form  $M(f_1, f_2, f_3, f_4, f_5, f_6)$  is said to be  $\xi$ - $D$ -conformally flat then  $f_3 = f_1 + 1, f_4 = f_6$ .

*Proof* Suppose the condition  $B(X, Y)\xi = 0$  holds in a  $(2n + 1)$ -dimensional generalized

$(\kappa, \mu)$  space form. We have from (2.1), (2.2) in (2.19)

$$\begin{aligned} 0 &= R(X, Y)\xi + \frac{1}{2(n-1)}\{S(X, \xi)Y - S(Y, \xi)X - S(X, \xi)\eta(Y)\xi + S(Y, \xi)\eta(X)\xi\} \\ &\quad + 2\{\eta(X)Y - \eta(Y)X\}. \end{aligned} \quad (3.5)$$

Using (2.7) and (2.15) we calculate

$$0 = \frac{f_3 - f_1 - 1}{n-1}\{\eta(Y)X - \eta(X)Y\} + (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\} \quad (3.6)$$

because of  $\{\eta(Y)X - \eta(X)Y\} \neq 0$  we must have  $f_3 - f_1 - 1 = 0$  and  $f_4 - f_6 = 0$ . Hence we obtain our proof.  $\square$

From Definition 2.3 we can state our next theorem.

**Theorem 3.2** *If a  $(2n+1)$ -dimensional generalized  $(\kappa, \mu)$  space form  $M(f_1, f_2, f_3, f_4, f_5, f_6)$  is  $\phi$ -D-conformally flat then Ricci tensor reduces to the form*

$$S(Y, Z) = \alpha g(Y, Z) + \beta \eta(Y)\eta(Z) + \gamma g(hY, Z) \quad (3.7)$$

under the condition  $Tr.\phi = 0$ , and  $\mu = 0$ , where  $\alpha, \beta, \gamma$  are constants.

*Proof* Let  $M(f_1, f_2, f_3, f_4, f_5, f_6)$  be a  $(2n+1)$ -dimensional generalized  $(\kappa, \mu)$  space form. Suppose  $M$  satisfies  $g(B(\phi X, \phi Y)\phi Z, \phi W) = 0$  then from (2.1), (2.19) we obtain

$$\begin{aligned} 0 &= g(R(\phi X, \phi Y)\phi Z, \phi W) \\ &\quad + \frac{1}{2(n-1)}\{S(\phi X, \phi Z)g(\phi Y, \phi W) - S(\phi Y, \phi Z)g(\phi X, \phi W) \\ &\quad + S(\phi Y, \phi W)g(\phi X, \phi Z) - S(\phi X, \phi W)g(\phi Y, \phi Z)\} \\ &\quad - \frac{k-2}{2(n-1)}\{g(\phi X, \phi Z)g(\phi Y, \phi W) - g(\phi Y, \phi Z)g(\phi X, \phi W)\} \end{aligned} \quad (3.8)$$

In view of (2.22) and (3.8) and having few steps of calculations we get

$$\begin{aligned} 0 &= f_1\{g(Y, Z)g(X, W) - g(Y, Z)\eta(X)\eta(W) - g(X, W)\eta(Y)\eta(Z) \\ &\quad - g(X, Z)g(Y, W) - g(Y, W)\eta(X)\eta(Z) + g(X, Z)\eta(Y)\eta(W)\} \\ &\quad + f_2\{g(X, \phi Z)g(\phi Y, W) - g(\phi Y, Z)g(X, \phi W) + 2g(X, \phi Y)g(\phi Z, W)\} \\ &\quad + f_4\{-g(Y, Z)g(hX, W) + \eta(Y)\eta(Z)g(hX, W) + g(X, Z)g(hY, W) \\ &\quad - \eta(X)\eta(Z)g(hY, W) - g(hY, Z)g(X, W) + \eta(X)\eta(W)g(hY, Z) + g(Y, W)g(hX, Z) \\ &\quad - g(hX, Z)\eta(Y)\eta(W)\} \\ &\quad + f_5\{g(hY, Z)g(hX, W) - g(hX, Z)g(hY, W) + g(hX, \phi Z)g(hY, \phi W) \\ &\quad - g(hY, Z)g(hX, W)\} + \frac{1}{2(n-1)}\{S(X, Z)g(Y, W) - S(X, Z)\eta(Y)\eta(W) \end{aligned} \quad (3.9)$$

$$\begin{aligned}
& -2n(f_1 - f_3)g(Y, W)\eta(X)\eta(Z) - S(Y, Z)g(X, W) + S(Y, Z)\eta(X)\eta(W) \\
& + 2n(f_1 - f_3)g(X, W)\eta(Y)\eta(Z) + S(Y, W)g(X, Z) - S(Y, W)\eta(X)\eta(Z) \\
& - 2n(f_1 - f_3)g(X, Z)\eta(Y)\eta(W) - S(X, W)g(Y, Z) + S(X, W)\eta(Y)\eta(Z) \\
& + 2n(f_1 - f_3)g(Y, Z)\eta(X)\eta(W) \} + \frac{(2n-1)f_4 - f_6}{2(n-1)} \{ -g(hX, Z)g(Y, W) \\
& + g(hX, Z)\eta(Y)\eta(W) + g(hY, Z)g(X, W) \\
& - g(hY, Z)\eta(X)\eta(W) - g(hY, Z)g(X, Z) + g(hY, Z)\eta(X)\eta(Z) \\
& + g(hX, W)g(Y, Z) - g(hX, W)\eta(Y)\eta(Z) \} \\
& - \frac{k-2}{2(n-1)} \{ g(X, Z)g(Y, W) - g(X, Z)\eta(Y)\eta(W) - g(Y, W)\eta(X)\eta(Z) \\
& - g(Y, Z)g(X, W) + g(Y, Z)\eta(X)\eta(W) + g(X, W)\eta(Y)\eta(Z) \}
\end{aligned}$$

Let  $\{e_i : i = 1, 2, \dots, 2n+1\}$  be an orthonormal basis of the tangent space at any point of the manifold. Putting  $X = W = e_i$  in (3.14) and taking summation over  $i$ ,  $1 \leq i \leq 2n+1$ , we get

$$\begin{aligned}
0 &= (2n-1)f_1\{g(Y, Z) - \eta(Y)\eta(Z)\} + f_2\{3g(\phi Y, \phi Z) - g(\phi Y, Z)Tr\phi\} \\
&\quad - (2n-1)f_4g(hY, Z) + \frac{1}{2(n-1)}[-2(n-1)S(Y, Z) - S(Z, \xi)\eta(Y) \\
&\quad + \{2n(2n-1)(f_1 - f_3) + r\}\eta(Y)\eta(Z) - S(Y, \xi)\eta(Z) \\
&\quad + \{2n(2n-1)(f_1 - f_3) - r\}g(Y, Z)] + \frac{r+2}{2(n-1)}[g(Y, Z) - \eta(Y)\eta(Z)] \\
&\quad + 0\frac{[(2n-1)f_4 - f_6](2n-1)}{2(n-1)}g(hY, Z)
\end{aligned} \tag{3.10}$$

By using (2.3) and (2.14) we obtain

$$\begin{aligned}
S(Y, Z) &= \frac{[(2n^2 - 2n + 1) + 3(n-1)f_2 - nf_3 + 1]}{n-1}g(Y, Z) \\
&\quad + \frac{[n(3-2n)f_3 - f_1 - 3(n-1)f_2 - 1]}{n-1}\eta(Y)\eta(Z) \\
&\quad + \frac{(f_4 - f_6)(2n-1)}{2(n-1)}g(hY, Z).
\end{aligned} \tag{3.11}$$

Assuming

$$\begin{aligned}
\alpha &= \frac{[(2n^2 - 2n + 1) + 3(n-1)f_2 - nf_3 + 1]}{n-1}, \\
\beta &= \frac{[n(3-2n)f_3 - f_1 - 3(n-1)f_2 - 1]}{n-1}, \\
\gamma &= \frac{(f_4 - f_6)(2n-1)}{2(n-1)}.
\end{aligned}$$

Hence we arrive at our proposed result.  $\square$

**Theorem 3.3** *If a  $(2n+1)$ -dimensional generalized  $(\kappa, \mu)$  space form  $M(f_1, f_2, f_3, f_4, f_5, f_6)$*

satisfies the condition  $B(\xi, X)S = 0$ , then the scalar curvature is given by

$$r = 2n(f_1 - f_3)(2n + 1) - 2n(f_4 - f_6)(f_1 - f_3 - 1)[(2n - 1)f_4 - f_6]. \quad (3.12)$$

*Proof* Let  $M(f_1, f_2, f_3, f_4, f_5, f_6)$  be a  $(2n + 1)$ -dimensional generalized  $(\kappa, \mu)$  space form. we suppose that  $M$  satisfies the condition  $(B(\xi, X)S)(U, V) = 0$ , where  $S$  is the Ricci tensor. Then we get

$$S(B(\xi, X)U, V) + S(U, B(\xi, X)V) = 0. \quad (3.13)$$

From equation (2.19) we can write after having few steps of calculations

$$\begin{aligned} B(\xi, Y)Z &= (f_1 - f_3 - \frac{1}{n-1})\{g(Y, Z)\xi - \eta(Z)Y\} + (f_4 - f_6)\{g(hY, Z)\xi - \eta(Z)hY\} \\ &\quad + \frac{1}{2(n-1)}\{2n(f_1 - f_3)[Y - \eta(Y)\xi]\eta(Z) \\ &\quad - [g(Y, Z) - \eta(Y)\eta(Z)]Q\xi\}. \end{aligned} \quad (3.14)$$

Similarly replacing  $Y$  with  $X$  and  $Z$  with  $U$  we obtain

$$\begin{aligned} B(\xi, X)U &= (f_1 - f_3 - \frac{1}{n-1})\{g(X, U)\xi - \eta(U)X\} + (f_4 - f_6)\{g(hX, U)\xi - \eta(U)hX\} \\ &\quad + \frac{1}{2(n-1)}\{2n(f_1 - f_3)[X - \eta(X)\xi]\eta(U) \\ &\quad - [g(X, U) - \eta(X)\eta(U)]Q\xi\}. \end{aligned} \quad (3.15)$$

Putting equations (3.14), (3.15) in (3.13) and replacing  $V$  with  $\xi$  we infer

$$\begin{aligned} 0 &= (f_1 - f_3 - \frac{1}{n-1})\{2n(f_1 - f_3)g(X, U) - S(X, \xi)\eta(U)\} \\ &\quad + \frac{n(f_1 - f_3)}{n-1}[\{S(X, \xi) - 2n(f_1 - f_3)\eta(X)\}\eta(U) - 2n(f_1 - f_3)\{g(X, U) - \eta(x)\eta(U)\}] \\ &\quad + (f_1 - f_3 - \frac{1}{n-1})\{2n(f_1 - f_3)\eta(X)\eta(U) - S(X, U)\} \\ &\quad + \frac{n(f_1 - f_3)}{n-1}[S(X, U) - 2n(f_1 - f_3)\{g(X, U) - \eta(x)\eta(U)\}] \\ &\quad + (f_4 - f_6)\{2n(f_1 - f_3)g(hX, U) - S(hX, U)\}. \end{aligned} \quad (3.16)$$

Using equations (2.1), (2.2), (2.12) we obtain

$$S(X, U) = 2n(f_1 - f_3)g(X, U) + \frac{1}{a}[(f_4 - f_6)\{2n(f_1 - f_3)g(hX, U) - S(hX, U)\}], \quad (3.17)$$

where  $a = f_1 - f_3 - \frac{2}{n-1}$ . Again taking the orthonormal frame field at any point of the manifold and contracting over  $X$  and  $U$  we get from above equation

$$r = 2n(f_1 - f_3)(2n + 1) - 2n(f_4 - f_6)(f_1 - f_3 - 1)[(2n - 1)f_4 - f_6]. \quad (3.18)$$

Hence we get the result.  $\square$

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## Spectrum of $(k, r)$ - Regular Hypergraphs

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**Abstract:** We present a spectral theory of uniform, regular and linear hypergraph. The main result are the nature of the eigenvalues of  $(k, r)$  - regular linear hypergraph and the relation between its dual and line graph. We also discuss some properties of Laplacian spectrum of a  $(k, r)$  - regular hypergraphs.

**Key Words:** Hypergraph, spectrum, dual of hypergraph, line graph, Laplacian spectrum, Smarandachely linear.

**AMS(2010):** 05C12.

### §1. Introduction

The spectral graph theory [2, 3] is the study of the relation between eigenvalues and eigenvectors of certain associated matrices of a graph and its combinatorial properties. There is some relation between the size of the eigenvalues and maximum degree of the graph [8, 1]. Connection between spectral characteristics of a graph and other graph theoretic parameters is a well explored area.

A graph structure is obtained when a non empty set (set of vertices) and a subset of all unordered pairs of elements of the set of vertices (this subset is called the set of edges) are taken together. When the set of unordered pairs is replaced by order pairs we get directed graphs. For a regular graph  $G$ , eigenvalues of the adjacency matrix are well studied. The second largest eigenvalue of the adjacency matrix is related to quantities such as diameter [6], chromatic number [4] etc. The regular graph with small non trivial eigenvalue can be used as good expanders and superconductors in communication network [12].

From the point of view of applications in social network and allied disciplines a more general structure is very useful. This structure, called *hypergraph* is obtained by taking a subset of the set of all proper subsets of the given set. The elements of the second set are called hyperedges. A hypergraph is denoted by  $H$ . For basic ideas and definitions on hypergraph readers may refer the text by [2], Chung [6] and Wen - Ching et al. [11] proposed the operator attached to a regular and uniform hypergraph and obtained some estimate of the eigenvalues of the operation.

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<sup>1</sup>Received October 05, 2016, Accepted May 18, 2017.

K. Feng and W. C. Li [8] studied the growth of the second largest eigenvalue and distribution of the eigenvalues of the adjacency matrix attached to a regular hypergraph.

In the next section we go through the main definitions and important results needed to understand the concepts included in the subsequent sections.

## §2. Preliminaries

We now discuss some basic terminology of hypergraphs, which we require in the sequel. A hypergraph  $H$  is a pair  $(X, E)$  where  $X = \{v_1, v_2, \dots, v_n\}$  is a finite set and  $E = \{E_1, E_2, \dots, E_m\}$  of subsets of  $X$ , such that  $E_j \neq \emptyset$ ,  $j = 1, 2, 3 \dots m$  and  $\bigcup_{i=1}^m E_i = X$ . The elements of  $X$  are called vertices or hypervertices and the elements of the sets  $\{E_1, E_2, \dots, E_m\}$  are called hyperedges of  $H$ . The cardinality of  $X$  is called the *order* of  $H$  and cardinality of  $E$  is called the *size* of  $H$ . In a hypergraph two vertices are adjacent if there is a hyperedge that containing both of these vertices. Two hyperedges are said to be adjacent if  $E_i \cap E_j \neq \emptyset$ . A *simple hypergraph* is a hypergraph such that  $E_i \subset E_j \Rightarrow i = j$ . A hypergraph  $H$  is said to be  $k$  - uniform for an integer  $k \geq 2$ , if for all  $E_i$  in  $E$ ,  $|E_i| = k$  for all  $i$ .

A hypergraph  $H$  is said to be *Smarandachely linear* respect to a subset  $E' \subset E(H)$  if  $|E_i \cap E_j| \leq 1$  for all  $E_i, E_j \in E'$  but  $|E_i \cap E_j| > 1$  if  $E_i, E_j \in E \setminus E'$ . Particularly, a hypergraph  $H$  is said to be *linear* if  $|E_i \cap E_j| \leq 1$  for all  $i \neq j$ , i.e., a Smarandachely linear in case of  $E' = E$ . A hypergraph in which all vertices have the same degree is said to be regular. An  $r$  - *regular hypergraph* is a hypergraph with  $d(v_i) = r$  for all  $i \leq n$  and an  $(r, k)$  - *regular hypergraph* is a hypergraph which is  $r$  - regular and  $k$  - uniform. The following results are important.

**Theorem 2.1([4])** *Let  $T$  be a real  $n \times n$  matrix with non negative entries and irreducible then there exists a unique positive real number  $\theta_0$  with the following properties:*

- (1) *there is a real number  $x_0 > 0$  with  $Tx_0 = \theta_0 x_0$ ;*
- (2) *any non negative right or left eigenvector of  $T$  has eigenvalue  $\theta_0$ . Suppose  $t \in R$  and  $x \geq 0$ ,  $x \neq 0$ . If  $Tx \leq tx$ , then  $x \geq 0$  and  $t \geq \theta_0$  and moreover  $t = \theta_0$  iff  $Tx = tx$ . If  $Tx \geq tx$ , then  $t \leq \theta_0$  and moreover  $t = \theta_0$  iff  $Tx = tx$ .*

**Theorem 2.2([4])** *Consider two sequence of real numbers  $\theta_1 \geq \theta_2 \geq \dots \geq \theta_n$  and  $\eta_1 \geq \eta_2 \geq \dots \geq \eta_m$  with  $m < n$ . The second is said to interlace the first one whenever  $\theta_i \geq \eta_{n-m+i}$  for  $i = 1, 2, \dots, m$ .*

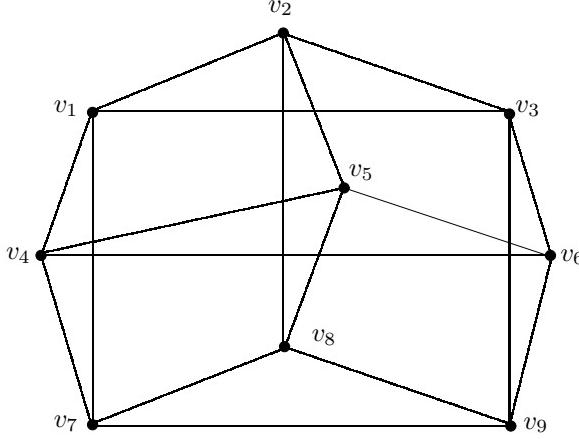
**Theorem 2.3([4])** *Let  $C$  be the quotient matrix of a symmetric matrix  $A$  whose rows and columns are partitioned according to the partitioning  $\{x_1, x_2, \dots, x_m\}$  then the eigenvalues of  $C$  interlace the eigenvalues of  $A$ .*

Let  $A$  be a real symmetric matrix and  $u$  be a non zero vector. The *Rayleigh Quotient* of  $u$  [2] with respect to  $A$  is defined as  $\frac{u^T A u}{u^T u}$ . The *dual* of the hypergraph  $H(X, E)$  [?] is a hypergraph  $H^* = H(X^*, E^*)$  where  $X^* = \{e_1, e_2, \dots, e_m\}$  corresponding to the edges of  $H$  and  $E^* = \{X_1, X_2, \dots, X_n\}$  where  $X_i = \{e_j : x_i \in E_j \text{ in } H\}$ . Also  $(H^*)^* = H$ . Given a hypergraph  $H = (X, E)$ , where  $X = \{E_1, E_2, \dots, E_m\}$ . Its *representative graph* or *line graph*

$\mathcal{L}(H)$ , is a graph whose vertices are points  $\{e_1, e_2, \dots, e_m\}$  representing the edges of  $H$  and the vertices  $e_i$  and  $e_j$  being adjacent if and only if  $E_i \cap E_j \neq \emptyset$ .

**Theorem 2.4([2])** *The dual of a linear hypergraph is also linear.*

*Proof* Given  $H$  is linear. Suppose  $H^*$  is not linear. Let  $X_1$  and  $X_2$  in  $H^*$  intersect at two distinct points  $e_1$  and  $e_2$ . Hence  $\{e_1, e_2\} \subset E_1$  and  $E_2$ . Therefore  $|E_1 \cap E_2| \geq 2$ , which contradicts  $|E_i \cap E_j| \leq 1$ . So  $H^*$  is linear.  $\square$



**Figure 1** An example of (2,3)- regular and linear hypergraph on 9 vertices

Now we define the *adjacency matrix* [1] of a hypergraph  $H$ . Adjacency matrix of  $H$  is denoted by  $A_H = (a_{ij})$ , where

$$a_{ij} = \begin{cases} |\{E_k \in E : \{v_i, v_j\} \subseteq E_k\}| & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of the adjacency matrix  $A_H$  is called the eigenvalues of  $H$ . Since  $A_H$  is a real symmetric matrix, all the eigenvalues are real. The spectrum of  $H$  is the set of all eigenvalues of  $A_H$  together with their multiplicities. Spectrum of  $H$  is denoted by  $Spec(H)$  or  $Spec(A_H)$ . Let  $\lambda_1, \lambda_2, \dots, \lambda_s$  be distinct eigenvalues of  $H$  with multiplicities  $m_1, m_2, \dots, m_s$  then we write

$$Spec(H) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_s \\ m_1 & m_2 & \dots & m_s \end{pmatrix}$$

In the next section we derive some properties of the spectrum and Laplacian spectrum of  $(r, k)$ -regular hypergraphs. Also discuss the relation between the Line graph and dual of 2-regular hypergraphs.

### §3. Spectrum of $(r, k)$ - Regular Hypergraphs

We know that the sum of the entries in each row and column of the adjacency matrix of an

$(r, k)$  - regular hypergraph  $H$  is  $\theta = r(k - 1)$ . Thus  $\theta$  is an eigenvalue of  $A(H)$  [9, 11].

**Theorem 3.1** *Let  $H$  be an  $(r, k)$  - regular linear hypergraph then*

- (1)  $\theta = r(k - 1)$  is an eigen value of  $H$ ;
- (2) if  $H$  is connected, then multiplicity of  $\theta$  is one;
- (3) for any eigenvalue  $\lambda$  of  $H$  we have  $|\lambda| \leq \theta$ .

*Proof* (1) Let  $A_H = A$  be the adjacency matrix of  $H$ . Also let  $u = (1, 1, 1, \dots, 1)^t$ . Since  $H$  is  $k$ -uniform, each edge has exactly  $k$  vertices. ie  $|E_j| \leq k$  for  $i \leq m$ . Since  $H$  is  $r$ - regular each vertex  $x_i$  belongs to exactly  $r$  hyperedge. Let  $v_i \in E_1, E_2, \dots, E_k$ . Then  $k$  hyper edge consist of  $rk$  vertices. So different pair of vertices with  $x_i$  as one factor is  $rk - r = r(k - 1)$ . Let  $\theta = r(k - 1)$ . Then  $Au = \theta u$  ie  $\theta$  is an eigenvalue of  $H$ .

(2) let  $z = (z_1, z_2, \dots, z_n)^t$ ,  $z \neq 0$  be such that  $Az = \theta z$  Let  $z_i$  be the entry of  $z$  having the largest absolute value  $(Az)_i = \theta z_i$

$$\sum_{j=1}^m a_{ij} z_j = \theta z_i.$$

We have each vertex  $x_i$  is associated with  $r(k - 1)$  other vertices through a hyperedge. By the maximality property of  $z_i$ ,  $z_j = z_i$  for all these vertices. Since  $H$  is connected all the vertices of  $z$  are equal. ie  $z$  is a multiple of  $u = (1, 1, \dots, 1)^t$ . Therefore the space of eigenvector associated with the eigen value  $\theta$  has dimension one. So the multiplicity of  $\theta$  is one.

(3) Suppose  $Ay = \lambda y$ ,  $y \neq 0$ . Let  $y_i$  denote the entry of  $y$  which is the largest in absolute value. Then,

$$(Ay)_i = \lambda y_i$$

implies that

$$\sum_{j=1}^n a_{ij} y_j = \lambda y_i.$$

Thus,

$$\begin{aligned} |\lambda y_i| &= \left| \sum_{j=1}^n a_{ij} y_j \right| \leq \sum_{j=1}^n |a_{ij}| |y_j| \\ &\leq |y_i| \sum_{j=1}^n a_{ij} = \theta |y_i|, \end{aligned}$$

i.e.,

$$|\lambda| \leq \theta.$$

□

**Proposition 3.2** *Let  $H$  be a  $k$  - uniform hypergraph with largest eigenvalue  $\lambda_1$ . If  $H$  is regular of degree  $r$ , then  $\lambda_1 = \theta$  where  $\theta = r(k - 1)$ . Otherwise  $(k - 1)\delta_{\min} \leq \bar{\delta} \leq \lambda_1 \leq (k - 1)\delta_{\max}$ , where  $\delta_{\min}, \delta_{\max}$  and  $\bar{\delta}$  are the minimum, maximum and average degree respectively.*

*Proof* Let  $\mathbf{1}$  be the vector with all entries equal to 1. Then  $A\mathbf{1} \leq (k-1)\delta_{\max}\mathbf{1}$ . By Theorem

2.1,  $\lambda_1 \leq \theta_{max}$  where  $\theta_{max} = (k - 1)\delta_{max}$  and equality happens if and only if  $A\mathbf{1} = \lambda_1\mathbf{1}$ , ie if and only if  $\lambda_1 = r(k - 1)$  where  $r$  is the degree of the vertices. Consider the partition of the vertex set consisting of a single part. By Theorem 2.3 we have  $\bar{\delta} \leq \lambda_1$ . Equality happens if and only if  $H$  is regular.  $\square$

**Proposition 3.3** *Let  $H$  be a linear hypergraph with eigenvalues  $\theta = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ , then the following are equivalent.*

- (1)  $H$  is an  $(r, k)$  - regular hypergraph;
- (2)  $AJ = \theta J$  where  $\theta = r(k - 1)$  and  $J$  is an  $n \times n$  matrix with all entries equal to 1;
- (3)  $\sum_{i=1}^n \lambda_i^2 = \theta n$ .

*Proof* The statements (1) and (2) are equivalent. In order to complete the theorem, we prove that that (1) and (3) are equivalent. First assume  $H$  is an  $(r, k)$  - regular hypergraph. Then

$$\sum_{i=1}^n \lambda_i^2 = \text{tr}(A^2) = \theta n.$$

Conversely, assume (3) is true. Then

$$\frac{1}{n} \sum_{i=1}^n \lambda_i^2 = \theta = r(k - 1) = \lambda_1.$$

By Proposition 3.2,  $H$  is regular.  $\square$

**Theorem 3.4** *The dual  $H^*$  of a  $(2, k)$  - regular hypergraph  $H$  is  $k$  - regular. Hence  $k$  is an eigenvalue of  $H^*$ .*

*Proof* Let  $H$  be a  $(2, k)$  - regular hypergraph. We know that  $d(v_i) = 2$  for  $i \leq n$  and  $|E_j| = k$ , for  $j \leq m$ .  $X_i = \{e_j/x_i \in E_j \text{ in } H\}$ . Each  $x_i$  belongs to exactly two edges in  $H$ . So  $|x_j| \leq 2$  for  $j \leq n$ . Since  $|E_j| = k$ , each  $e_i$  is adjacent to exactly  $k$ ,  $e'_j$ s hence  $d(e_j) = k$ . Therefore  $H^*$  is a  $k$  - regular simple graph. Hence  $k$  is an eigenvalue of  $H^*$ .  $\square$

**Theorem 3.5** *Let  $H$  be a  $(2, k)$  - regular linear hypergraph. Its line graph  $\mathcal{L} \cong H^*$ .*

*Proof* By Theorem 3.4 ,  $H^*$  is a  $k$  - regular simple graph. Let  $\{x_1, x_2, \dots, x_m\}$  be the vertices of  $\mathcal{L}$ . In  $H$ , each edge  $E_j$  has  $k$  - vertices. Since the vertex  $x_j$  is adjacent to  $k$  other vertices,  $d(x_j) = k$  for  $j \leq m$ .  $\mathcal{L}$  is a  $k$  - regular simple graph. Hence  $\mathcal{L}$  and  $H^*$  have same number of vertices and edges. Also incidence relation is preserved. Therefore  $L(H) \cong H^*$ .  $\square$

#### §4. Laplacian Spectrum of $(r, k)$ - Regular Hypergraphs

We define the itLaplacian degree of a vertex  $v_i \in X(H)$  as  $\delta_l(v_i) = \sum_{j=1}^n a_{ij}$ . We say that the hypergraph  $H$  is Laplacian regular of degree  $\delta_l$  if any vertex  $v \in X(H)$  has Laplacian degree  $\delta_l(v) = \delta_l$ . If  $H$  is a simple graph, then  $\delta_l(v_i) = \delta(v_i)$ . The Laplacian matrix of a hypergraph  $H$

is denoted by  $L = L(H)$  and is defined as  $L = D - A$  where  $D = \text{diag}\{\delta_l(v_1), \delta_l(v_2), \dots, \delta_l(v_n)\}$ . The matrix  $L$  is symmetric and positive semi definite. All the eigen values are real and non negative. The smallest eigen value is 0 and the corresponding eigen vector is  $j = (1, 1, \dots, 1)$ . Moreover the multiplicity is the number of connected components of  $H$ . Second smallest and largest Laplacian eigen values and parameters related are studied by Rodrigues [10]. The eigen values of  $L$  are denoted by  $0 = \mu_1 < \mu_2 < \dots < \mu_b$  and their multiplicities  $m_1, m_2, m_3, \dots, m_b$ . Thus the Laplacian spectrum of  $H$  and Laplacian degree of its vertices are related by

$$\sum_{l=1}^b m_l \mu_l = \text{tr}(L(H)) = \sum_{i=1}^n \delta_l(v_i). \quad (4.1)$$

If  $H$  is a regular hypergraph of degree  $\delta(l)$ , then  $L = D - A = \delta_l I - A$ . Thus the eigenvalues of  $A$  and  $L$  are related by  $\mu_i = \delta_i - \lambda_i$ ,  $i = 1, 2, \dots, b$ .

The second smallest eigenvalue of the graph gives the most important information contained in the spectrum. This eigenvalue is called the *algebraic connectivity* and is related to several graph invariants and imposes reasonably good bounds on the value of several parameters of graphs which are very hard to compute. The concept of algebraic connectivity was introduced by Fiedler [7]. Also  $\mu_2 \geq 0$  with equality if and only if  $H$  is disconnected. Algebraic connectivity is monotone. It does not decrease when edges are added to the graph.

**Theorem 4.1** *Let  $H$  be a  $k$ -uniform hypergraph. Also let  $\mu_2$  be the algebraic connectivity of  $H$ . For any non adjacent vertices  $s$  and  $t$  in  $H$  we have*

$$\mu_2(H) \leq \frac{k-1}{2}(\deg(s) + \deg(t)).$$

*Proof* We have,

$$\mu_2(H) = \min_u \left\{ \frac{\langle Lu, u \rangle}{\langle u, u \rangle} : \langle u, \mathbf{1} \rangle = 0 \right\}.$$

The vector  $u$  is defined by

$$u_x = \begin{cases} 1 & \text{if } x = s \\ -1 & \text{if } x = t \\ 0 & \text{otherwise.} \end{cases}$$

Clearly  $\langle u, \mathbf{1} \rangle = 0$  and by definition,

$$\begin{aligned} \mu_2(H) &\leq \frac{\langle Lu, u \rangle}{\langle u, u \rangle} = \frac{\sum_{xy \in E_j} a_{xy} (u_x - u_y)^2}{\sum_{x \in V} u_x^2} \\ &= \frac{(k-1)(\sum_x a_{xs} + \sum_x a_{tx})}{2} = \frac{(k-1)}{2} (\deg(s) + \deg(t)). \end{aligned} \quad \square$$

Let  $G$  be the weighted graph on the same vertex set  $X$ . Two vertices  $x$  and  $y$  are adjacent in  $X$  if they are adjacent in  $H$  also. The edge weight of  $xy$  is equal to  $(a_{ij})$ , the number of edges in  $H$  containing both  $x$  and  $y$ . Clearly  $L(G) = L(H)$ . Rodriguez [10] obtain the result

from Fiedler [7] on weighted graph as

$$\mu_2 = 2n \min \left\{ \frac{\sum_{xy \in E_j} a_{xy}(u_x - u_y)^2}{\sum_{x \in X} \sum_{y \in X} (u_x - u_y)^2} : u \neq \alpha \mathbf{1} \text{ for } \alpha \in R \right\}, \quad (4.2)$$

$$\mu_b = 2n \max \left\{ \frac{\sum_{xy \in E_j} a_{xy}(u_x - u_y)^2}{\sum_{x \in X} \sum_{y \in X} (u_x - u_y)^2} : u \neq \alpha \mathbf{1} \text{ for } \alpha \in R \right\}. \quad (4.3)$$

This eigenvalues  $\mu_2$  and  $\mu_b$  are bounded in terms of maximum and minimum degree of  $H$ . For any vertex  $x$ ,  $e_x$  denote the corresponding unit vector of the canonical basis of  $R^n$  by putting  $u = e_x$  in equations (4.2) and (4.3) we get,

$$\mu_2 \leq \frac{n}{n-1} \delta_x \leq \mu_b.$$

This leads to

$$\mu_2 \leq \frac{n}{n-1} \delta_{l \text{ min}} \leq \frac{n}{n-1} \delta_{l \text{ max}} \leq \mu_b.$$

**Theorem 4.2** *Let  $H_1$  and  $H_2$  be two edge disjoint hypergraphs on the same vertex set and  $H_1 \cup H_2$  be their union. Then*

$$\mu_2(H_1 \cup H_2) \geq \mu_2(H_1) + \mu_2(H_2) \geq \mu_2(H_1).$$

*Proof* Let  $L_1, L_2$  and  $L$  be the Laplace adjacency matrix of  $H_1, H_2$  and  $H = H_1 \cup H_2$  respectively. We have,

$$\begin{aligned} \mu_2(H) &= \min_u \left\{ \frac{\langle Lu, u \rangle}{\langle u, u \rangle} : \langle u, \mathbf{1} \rangle = 0 \right\} \\ &= \min_u \left\{ \frac{\langle (L_1 + L_2)u, u \rangle}{\langle u, u \rangle} : \langle u, \mathbf{1} \rangle = 0 \right\} \\ &= \min_u \left\{ \frac{\langle (L_1 u + L_2 u), u \rangle}{\langle u, u \rangle} : \langle u, \mathbf{1} \rangle = 0 \right\} \\ &\geq \min_u \left\{ \frac{\langle L_1 u, u \rangle}{\langle u, u \rangle} : \langle u, \mathbf{1} \rangle = 0 \right\} \\ &\quad + \min_u \left\{ \frac{\langle L_2 u, u \rangle}{\langle u, u \rangle} : \langle u, \mathbf{1} \rangle = 0 \right\} \\ &\geq \mu_1(H_1) + \mu_2(H_2) \geq \mu_1(H_1). \end{aligned} \quad \square$$

### Acknowledgments

A part of this research was done when the first author visited the Kerala School of Mathematics (KSOM) at Kozhikodu, Kerala in India. He is indebted to KSOM for research facilities given by the institute.

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## On the Spacelike Parallel Ruled Surfaces with Darboux Frame

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**Abstract:** In this study, the spacelike parallel ruled surfaces with Darboux frame are introduced in Minkowski 3-space. Then some characteristic properties of the spacelike parallel ruled surfaces with Darboux frame such as developability, the striction point and the distribution parameter are obtained in Minkowski 3-space.

**Key Words:** Ruled surface, parallel surface, Darboux frame.

**AMS(2010):** 53A05, 53A15.

### §1. Introduction

In differential geometry, ruled surface is a special type of surface which can be defined by choosing a curve and a line along that curve. The ruled surfaces are one of the easiest of all surfaces to parametrize. That surface was found and investigated by Gaspard Monge who established the partial differential equation that satisfies all ruled surface. V. Hlavaty [9] also investigated ruled surfaces which are formed by one parameter set of lines.

A surface and another surface which have constant distance with the reference surface along its surface normal have a relationship between their parametric representations. Such surfaces are called parallel surface [8]. By this definition, it is convenient to carry the points of a surface to the points of another surface. Since the curves are set of points, then the curves lying fully on a reference surface can be carry to another surface.

Another one of the most important subjects of the differential geometry is the Darboux frame which is a natural moving frame constructed on a surface. It is the version of the Frenet frame as applied to surface geometry. A Darboux frame exists on a surface in a Euclidean or non-Euclidean spaces. It is named after the French mathematician Jean Gaston Darboux, in the four volume collection of studies published between 1887 and 1896. Since that time, there have been many important repercussions of Darboux frame, having been examined for example in [2], [3].

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<sup>1</sup>Received September 26, 2016, Accepted May 20, 2017.

The Minkowski space  $E_1^3$  is the Euclidean space  $E^3$  provided with the Lorentzian inner product

$$\langle u, v \rangle = u_1v_1 + u_2v_2 - u_3v_3,$$

where

$$u = (u_1, u_2, u_3), \quad v = (v_1, v_2, v_3) \in E^3.$$

We say that a vector  $u$  in  $E_1^3$  is spacelike, lightlike or timelike if

$$\langle u, u \rangle > 0, \quad \langle u, u \rangle = 0 \text{ or } \langle u, u \rangle < 0,$$

respectively. The norm of the vector  $u \in E_1^3$  is defined by

$$\|u\| = \sqrt{|\langle u, u \rangle|}.$$

G.Y. Senturk and S. Yucel have taken into a consideration about the ruled surface with Darboux frame in  $E^3$  ([4]).

By making use of the paper of Unluturk et al. [10], we describe our general approach to compute the spacelike parallel ruled surfaces with Darboux frame, and give some theorems for these kinds of surfaces.

## §2. Preliminaries

A ruled surface  $M$  in  $\mathbb{R}^3$  is generated by a one-parameter family of straight lines which are called the rulings. The equation of the ruled surface can be written as

$$\varphi(s, v) = \alpha(s) + vX(s),$$

where  $(\alpha)$  is curve which is called the base curve of the ruled surface and  $(X)$  is called the unit direction vector of the straight line.

An unit direction vector of straight line  $X$  is stretched by the system  $\{T, g\}$ . So it can be written as

$$X = T \sin \phi + g \cos \phi,$$

where  $\phi$  is the angle between  $T$  and  $X$  vectors ([10]).

The striction point on the ruled surface is the foot of the common perpendicular line successive rulings on the main ruling. The set of the striction points of the ruled surface generates its striction curve. It is given by ([10])

$$c(s) = \alpha(s) - \frac{\langle \alpha_s, X_s \rangle}{\langle X_s, X_s \rangle} X(s).$$

**Theorem 2.1([7])** *If successive rulings intersect, the ruled surface is called developable. The unit tangent vector of the striction curve of a developable ruled surface is the unit vector with direction  $X$ .*

The distribution of the ruled surface is identified by

$$P_X = \frac{\det(\alpha_s, X, X_s)}{\langle X_s, X_s \rangle}.$$

**Theorem 2.2([7])** *The ruled surface is developable if and only if  $P_X = 0$ .*

The ruled surface is said to be a noncylindrical ruled surface provided that ([1])

$$\langle X_s, X_s \rangle \neq 0$$

**Theorem 2.3([1])** *Let  $M$  be a noncylindrical ruled surface and defined by its striction curve. The Gaussian curvature of  $M$  is given by its distribution parameter by*

$$K = -\frac{P_X^2}{(P_X^2 + v^2)^2}$$

**Definition 2.1([5])** *Let  $M$  and  $\overline{M}$  be two surfaces in Euclidean space. The function*

$$\begin{aligned} f : M &\rightarrow \overline{M} \\ P &\rightarrow f(P) = P + rN_P \end{aligned} \tag{2.1}$$

*is called the parallelization function between  $M$  and  $\overline{M}$  and furthermore  $\overline{M}$  is called parallel surface to  $M$ , where  $N$  is the unit normal vector on  $M$  and  $r$  is a given real number.*

**Theorem 2.4([5])** *Let  $M$  and  $\overline{M}$  be two parallel surfaces in Euclidean space and*

$$f : M \rightarrow \overline{M}$$

*be the parallelization function. Then for  $X \in \chi(M)$*

1.  $f_*(X) = X - rS(X)$
2.  $S^r(f_*(X)) = S(X)$
3.  $f$  preserves principal directions of curvature, that is

$$S^r(f_*(X)) = \frac{\kappa}{1 - r\kappa} f_*(X),$$

*where  $S^r$  is the shape operator on  $\overline{M}$ , and  $\kappa$  is a principal curvature of  $M$  at  $P$  in direction  $X$ .*

**Theorem 2.5([11])** *Let Darboux frame of curve  $\beta$  at  $f(\alpha(t_0)) = f(P)$  on  $\overline{M}$  be  $\{\overline{T}, \overline{g}, \overline{N}\}$ , then*

$$\begin{aligned} \overline{T} &= \frac{1}{v} [(1 - r\kappa_n)T - r\tau_g g], \\ \overline{g} &= \frac{1}{v} [(1 - r\kappa_n)g + r\tau_g T], \\ \overline{N} &= N \end{aligned} \tag{2.2}$$

**Definition 2.2([2])** Let  $x$  be a spacelike vector and  $y$  be a timelike vector in  $E_1^3$ . Then there is an unique real number  $\theta \geq 0$  such that  $\langle x, y \rangle = |x| |y| \sinh \theta$ . This number is called the Lorentzian timelike angle between the vectors  $x$  and  $y$ .

**Definition 2.3([2])** Let  $x$  and  $y$  be future pointing (or past pointing) timelike vectors in  $E_1^3$ . Then there is an unique real number  $\theta \geq 0$  such that  $\langle x, y \rangle = -|x| |y| \cosh \theta$ . This number is called the hyperbolic angle between the vectors  $x$  and  $y$ .

We denote by  $\{T, n, b\}$  the moving Frenet frame along the unit speed curve  $\alpha(s)$  in the Minkowski space  $E_1^3$ , the following Frenet formulae are given

$$\frac{d}{ds} \begin{bmatrix} T \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\varepsilon \kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ n \\ b \end{bmatrix}, \quad (2.3)$$

in [6], where  $\langle T, T \rangle = 1$ ,  $\langle n, n \rangle = \varepsilon = \pm 1$ ,  $\langle b, b \rangle = -1$ .

If the surface  $M$  is a spacelike surface, then the curve  $\alpha(s)$  lying on surface  $M$  is a spacelike curve. So, the relations between the frames can be given as follows ([6]):

$$\begin{bmatrix} T \\ g \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta \\ 0 & \sinh \theta & \cosh \theta \end{bmatrix} \begin{bmatrix} T \\ n \\ b \end{bmatrix}, \quad (2.4)$$

where  $\{T, g, N\}$  is Darboux frame.

Besides, the derivative formulae of the Darboux frame of  $\alpha(s)$  is given by ([6])

$$\frac{d}{ds} \begin{bmatrix} T \\ g \\ N \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ g \\ N \end{bmatrix}, \quad (2.5)$$

where  $\langle T, T \rangle = \langle g, g \rangle = 1$  and  $\langle N, N \rangle = -1$ .

**Theorem 2.6([2])** Let  $\alpha(s)$  be non-unit speed curve on surface  $M$  in  $E_1^3$ . The Darboux frame of curve  $\alpha(s)$  where  $\|\alpha'(s)\| = v$ , is  $\{T, g, N\}$ . Geodesic, normal curvatures and geodesic torsion of this curve-surface pair which are denoted by  $\kappa_g$ ,  $\kappa_n$ ,  $\tau_g$  respectively are defined as follows:

$$\begin{aligned} \kappa_g &= \frac{1}{v^2} \langle \alpha'', g \rangle, \\ \kappa_n &= \frac{1}{v^2} \langle \alpha'', N \rangle, \\ \tau_g &= -\frac{1}{v} \langle N', g \rangle \end{aligned} \quad (2.6)$$

### §3. On the Spacelike Parallel Ruled Surfaces with Darboux Frame

A spacelike parallel ruled surface can be given as in the following parametrization:

$$\bar{\varphi}(s, \bar{v}) = \bar{\alpha}(s) + \bar{v}\bar{X}(s), \quad (3.1)$$

where the curve  $\bar{\alpha}(s)$  is lying on  $\bar{\varphi}(s, \bar{v})$  is a spacelike curve.

Darboux frame is obtained by rotating Frenet frame around  $\bar{T}$  as far as  $\bar{\theta} = \bar{\theta}(s)$  while the hyperbolic angle  $\bar{\theta}$  is between the timelike unit vector  $\bar{b}$  and the timelike normal vector field  $\bar{N}$  of  $\bar{\varphi}(s, \bar{v})$ .

By Definitions 2.2 and 2.3, the spacelike unit vector  $\bar{g}$  and the timelike vector  $\bar{N}$  are written, in terms of  $\bar{\theta}$ , as:

$$\begin{aligned} \bar{g} &= \bar{n} \cosh \bar{\theta} + \bar{b} \sinh \bar{\theta}, \\ \bar{N} &= \bar{n} \sinh \bar{\theta} + \bar{b} \cosh \bar{\theta}. \end{aligned} \quad (3.2)$$

From the expression (3.2), the spacelike unit vector  $\bar{n}$  and timelike unit binormal vector  $\bar{b}$  are obtained

$$\begin{aligned} \bar{n} &= \bar{g} \cosh \bar{\theta} - \bar{N} \sinh \bar{\theta}, \\ \bar{b} &= -\bar{g} \sinh \bar{\theta} + \bar{N} \cosh \bar{\theta}. \end{aligned} \quad (3.3)$$

Taking into the expressions (3.2) and (3.3), Darboux derivative formulae of the spacelike parallel ruled surface can be found as

$$\frac{d}{ds} \begin{bmatrix} \bar{T} \\ \bar{g} \\ \bar{N} \end{bmatrix} = \begin{bmatrix} 0 & \bar{\kappa}_g & \bar{\kappa}_n \\ -\bar{\kappa}_g & 0 & \bar{\tau}_g \\ \bar{\kappa}_n & \bar{\tau}_g & 0 \end{bmatrix} \begin{bmatrix} \bar{T} \\ \bar{g} \\ \bar{N} \end{bmatrix}, \quad (3.4)$$

where

$$\bar{\kappa}_g = \bar{\kappa} \cosh \bar{\theta}, \quad \bar{\kappa}_n = -\bar{\kappa} \sinh \bar{\theta} \text{ and } \bar{\tau}_g = \bar{\tau} + \bar{\theta}' \quad (3.5)$$

Differentiating the expression (3.3) and using the matrix representation (3.4), we have the Frenet derivative formulae of the spacelike parallel ruled surface as:

$$\begin{aligned} \bar{T}' &= \bar{\kappa} \bar{n}, \\ \bar{n}' &= -\bar{\kappa} \bar{T} + \bar{\tau} \bar{b}, \\ \bar{b}' &= \bar{\tau} \bar{n}. \end{aligned} \quad (3.6)$$

**Theorem 3.1** Let  $\bar{M}$  be a spacelike parallel ruled surface and Darboux frame of curve  $\beta$  be

$\{\overline{T}, \overline{g}, \overline{N}\}$  at  $f(\alpha(t_0)) = f(P)$  on  $\overline{M}$ , then we have

$$\begin{aligned}\overline{T} &= \frac{1}{v} [(1 + r\kappa_n)T + r\tau_g g], \\ \overline{g} &= \frac{1}{v} [-(1 + r\kappa_n)g + r\tau_g T], \\ \overline{N} &= N.\end{aligned}\tag{3.7}$$

*Proof* From Theorem 2.4 and the matrix representation (2.5), tangent vector of  $(f \circ \alpha) = \beta$  of the curve at  $f(\alpha(t_0))$  on the spacelike parallel ruled surface  $\overline{M}$  is

$$\begin{aligned}\beta' &= \overline{T} = \frac{1}{v} [T + r(\kappa_n T + \tau_g g)] \\ &= \frac{1}{v} [(1 + r\kappa_n)T + r\tau_g g],\end{aligned}\tag{3.8}$$

where the norm of  $\beta'$  is

$$\|\beta'\| = \sqrt{(1 + r\kappa_n)^2 + r^2 \tau_g^2} = v.\tag{3.9}$$

Same as, we find  $\overline{g}$ . And also there is the equation  $\overline{N} = N$  between normal vectors of surfaces  $M$  and  $\overline{M}$ .  $\square$

**Theorem 3.2** Let  $\alpha$  be a regular curve on the surface  $M$ . Then the geodesic curvature, the normal curvature and the geodesic torsion of the curve  $(f \circ \alpha) = \beta$  are respectively;

$$\begin{aligned}\overline{\kappa}_g &= -\frac{\kappa_g}{v^3} + \frac{r}{v^3} [(1 + r\kappa_n)\tau_g^1 + r\tau_g \kappa'_n], \\ \overline{\kappa}_n &= \frac{1}{v^2} [\kappa_n + r(\kappa_n^2 + \tau_g^2)], \\ \overline{\tau}_g &= \frac{\tau_g}{v^2},\end{aligned}\tag{3.10}$$

at the point  $f(\alpha(t_0))$  on the spacelike parallel ruled surface  $\overline{M}$ .

*Proof* Because  $\beta$  spacelike curve is a non-unit speed curve, we use the Theorems 2.6, 2.4 and the equation (3.8), hence the following equation is obtained

$$\beta'' = (r\kappa'_n - r\tau_g \kappa_g)T + (\kappa_g(1 + r\kappa_n) + r\tau_g^1)g + (\kappa_n(1 + r\kappa_n) + r\tau_g^2)N\tag{3.11}$$

Using the expressions (2.6), (3.7) and (3.11), we find  $\overline{\kappa}_g$ ,  $\overline{\kappa}_n$ ,  $\overline{\tau}_g$  for spacelike parallel ruled surface  $\overline{M}$ .  $\square$

**Theorem 3.3** Let Frenet frame of the curve  $(f \circ \alpha) = \beta$  be  $\{\overline{T}, \overline{n}, \overline{b}\}$  at  $f(\alpha(t_0)) = f(P)$  on

the surface  $\overline{M}$ , then Frenet frame of the spacelike parallel ruled surface is as follows:

$$\begin{aligned}\overline{T} &= \frac{1}{v} [(1+r\kappa_n)T + r\tau_g g], \\ \overline{n} &= \frac{1}{v\sqrt{(\kappa_n)^2 - (\kappa_g)^2}} [-(r\tau_g\kappa_n)T + (1+r\kappa_n)\kappa_g g + (v\kappa_n)N], \\ \overline{b} &= \frac{1}{v^3\sqrt{(\kappa_n)^2 - (\kappa_g)^2}} [(v^2 r\tau_g\kappa_n)T - (v^2(1+r\kappa_n)\kappa_g)g - (v^3\kappa_g)N].\end{aligned}\tag{3.12}$$

*Proof* If we use the equations (3.8) and (3.11), the following equation is obtained

$$\beta' \wedge \beta'' = (r\tau_g\kappa_n v^2) T - ((1+r\kappa_n)\kappa_n v^2) g - (\kappa_g v^3) N.\tag{3.13}$$

The norm of the equation (3.13) is

$$\|\beta' \wedge \beta''\| = v^3 \sqrt{(\kappa_n)^2 - (\kappa_g)^2}.\tag{3.14}$$

By the equations (3.13) and (3.14), we obtain the expression (3.12).  $\square$

Let  $\overline{\phi}$  be the angle between direction vector  $\overline{X}$  and tangent vector  $\overline{T}$  at  $\overline{\alpha} \in \overline{\varphi}(s, \overline{v})$ . If we choose the direction vector  $\overline{X}$ , then we get

$$\overline{X} = \overline{T} \cos \overline{\phi} + \overline{g} \sin \overline{\phi},\tag{3.15}$$

where  $\|\overline{X}\| = 1$ .

Differentiating (3.15) and using (3.4), we find

$$\overline{X}' = -(\overline{\phi}' + \kappa_g) \sin \overline{\phi} \overline{T} + (\overline{\phi}' + \kappa_g) \cos \overline{\phi} \overline{g} + (\kappa_n \cos \overline{\phi} + \tau_g \sin \overline{\phi}) \overline{N}.\tag{3.16}$$

Holding  $\overline{v}$  =constant, we obtain the curve on the spacelike parallel ruled surface whose vector field as follows:

$$\begin{aligned}\overline{T}^* &= \overline{T} + \overline{v} \overline{X}' \\ &= (1-\overline{v})(\overline{\phi}' + \kappa_g) \sin \overline{\phi} \overline{T} + (\overline{\phi}' + \kappa_g) \cos \overline{\phi} \overline{g} + (\kappa_n \cos \overline{\phi} + \tau_g \sin \overline{\phi}) \overline{N}.\end{aligned}\tag{3.17}$$

Substituting the expression (2.6) into (3.5), we have

$$\begin{aligned}\overline{T}^* &= \left( 1 - \frac{\overline{v}}{v^3} (-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3 \overline{\phi}') \sin \overline{\phi} \right) \overline{T} \\ &\quad + \frac{\overline{v}}{v^3} \left( (-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3 \overline{\phi}')(\kappa_n + r(\kappa_n^2 + \tau_g^2) + v^2 \overline{\phi}') \cos \overline{\phi} \right) \overline{g} \\ &\quad + \frac{\overline{v}}{v^2} \left( (\kappa_n + r(\kappa_n^2 + \tau_g^2)) \cos \overline{\phi} + \tau_g \sin \overline{\phi} \right) \overline{N}.\end{aligned}$$

The distribution parameter of the spacelike parallel ruled surface is defined by

$$\bar{P}_{\bar{X}} = \frac{-\sin \bar{\phi}(\bar{\kappa}_n \cos \bar{\phi} + \bar{\tau}_g \sin \bar{\phi})}{(\bar{\phi}' + \bar{\kappa}_g)^2 - (\bar{\kappa}_n \cos \bar{\phi} + \bar{\tau}_g \sin \bar{\phi})^2}. \quad (3.18)$$

By the expression (3.10), we obtain  $\bar{P}_{\bar{X}}$  for spacelike parallel ruled surface as follows:

$$\bar{P}_{\bar{X}} = \frac{-v^4 \sin \bar{\phi} ((\kappa_n + r(\kappa_n^2 + \tau_g^2)) \cos \bar{\phi} + \tau_g \sin \bar{\phi})}{(-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3 \bar{\phi}')^2 + v^2 ((\kappa_n + r(\kappa_n^2 + \tau_g^2)) \cos \bar{\phi} + \tau_g \sin \bar{\phi})^2}.$$

**Theorem 3.4** *The spacelike parallel ruled surface with Darboux frame is developable surface if and only if*

$$\sin \bar{\phi}(\bar{\kappa}_n \cos \bar{\phi} + \bar{\tau}_g \sin \bar{\phi}) = 0. \quad (3.19)$$

*Proof* Supposing that the spacelike parallel ruled surface with Darboux frame is developable surface, then  $\bar{P}_{\bar{X}} = 0$ . In this case, let us study the following subcases related to the equation (3.19) vanishing:

(1) If  $\sin \bar{\phi} = 0$ , then from (3.15), we obtain  $\bar{X} = \bar{T} \cos \bar{\phi}$ . So  $\bar{T}^* = \bar{T}$ . It means that tangent plane is constant along the main ruling on the spacelike parallel ruled surface.

(2) If  $(\bar{\kappa}_n \cos \bar{\phi} + \bar{\tau}_g \sin \bar{\phi}) = 0$ , then from the equation (3.17), it is seen that the tangent plane and the normal vector of spacelike parallel ruled surface with Darboux frame are orthogonal vectors. Therefore the spacelike parallel ruled surface with Darboux frame is developable surface.

Conversely, if

$$\sin \bar{\phi}(\bar{\kappa}_n \cos \bar{\phi} + \bar{\tau}_g \sin \bar{\phi}) = 0,$$

then from (3.18),  $\bar{P}_{\bar{X}} = 0$ .

The striction curve of the spacelike parallel ruled surface with Darboux frame is calculated as follows:

$$\bar{c}(s) = \bar{\alpha}(s) + \frac{\sin \bar{\phi}(\bar{\phi}' + \bar{\kappa}_g)}{(\bar{\phi}' + \bar{\kappa}_g)^2 + (\bar{\kappa}_n \cos \bar{\phi} + \bar{\tau}_g \sin \bar{\phi})^2} \bar{X}$$

Using the expression (3.10), we find  $\bar{c}(s)$  for the spacelike parallel ruled surface as:

$$\bar{c}(s) = \bar{\alpha}(s) + \frac{-v^4 \sin \bar{\phi}(-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3 \bar{\phi}')}{(-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3 \bar{\phi}')^2 - v^2 ((\kappa_n + r(\kappa_n^2 + \tau_g^2)) \cos \bar{\phi} + \tau_g \sin \bar{\phi})^2} \bar{X} \quad (3.20)$$

□

**Theorem 3.5** *Let  $\bar{M}$  be a spacelike parallel ruled surface with Darboux frame as in (3.1). Then the shortest distance between the rulings of  $\bar{M}$  along the orthogonal trajectories is the distance measured equaled to the value:*

$$\bar{v} = \frac{-v^4 \sin \bar{\phi}(-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3 \bar{\phi}')}{(-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3 \bar{\phi}')^2 - v^2 ((\kappa_n + r(\kappa_n^2 + \tau_g^2)) \cos \bar{\phi} + \tau_g \sin \bar{\phi})^2}. \quad (3.21)$$

*Proof* From (3.17), we have

$$\overline{J}(\overline{v}) = \int_{s_1}^{s_2} \left( 1 - 2\overline{v}(\overline{\phi}' + \overline{\kappa}_g) \sin \overline{\phi} + \overline{v}^2 (\overline{\phi}' + \overline{\kappa}_g)^2 + \overline{v}^2 (\overline{\kappa}_n \cos \overline{\phi} + \overline{\tau}_g \sin \overline{\phi})^2 \right)^{\frac{1}{2}} ds, \quad (3.22)$$

where  $s_1 < s_2$ .

Differentiating the expression (3.22) according to the parameter  $\overline{v}$  which gives the minimal value of  $\overline{J}(\overline{v})$ , we get

$$\overline{v} = \frac{\sin \overline{\phi}(\overline{\phi}' + \overline{\kappa}_g)}{(\overline{\phi}' + \overline{\kappa}_g)^2 + (\overline{\kappa}_n \cos \overline{\phi} + \overline{\tau}_g \sin \overline{\phi})^2}. \quad (3.23)$$

Using the expression (3.10) in (3.23), the parameter  $\overline{v}$  turns into the equation (3.21).  $\square$

**Theorem 3.6** Let  $\overline{M}$  be a spacelike parallel ruled surface with Darboux frame as in (3.1), the absolute value of Gauss curvature  $K$  of the spacelike parallel ruled surface  $\overline{M}$  along a ruling takes the maximum value at the striction point on that ruling.

*Proof* Calculating Gauss curvature of the spacelike parallel ruled surface with Darboux frame, we get

$$K(s, \overline{v}) = \frac{(\overline{\kappa}_n \cos \overline{\phi} + \overline{\tau}_g \sin \overline{\phi})^2 \sin^2 \overline{\phi}}{(1 - 2\overline{v}(\overline{\phi}' + \overline{\kappa}_g) \sin \overline{\phi} + \overline{v}^2 (\overline{\phi}' + \overline{\kappa}_g)^2 + \overline{v}^2 (\overline{\kappa}_n \cos \overline{\phi} + \overline{\tau}_g \sin \overline{\phi})^2 - \cos^2 \overline{\phi})^2}. \quad (3.24)$$

Differentiating the equation (3.24) with respect to  $\overline{v}$ , we have

$$\overline{v} = \frac{\sin \overline{\phi}(\overline{\phi}' + \overline{\kappa}_g)}{(\overline{\phi}' + \overline{\kappa}_g)^2 + (\overline{\kappa}_n \cos \overline{\phi} + \overline{\tau}_g \sin \overline{\phi})^2}.$$

Therefore, the absolute value of Gauss curvature  $K$  of the spacelike parallel ruled surface  $\overline{M}$  along a ruling takes the maximum value at the striction point on that ruling.  $\square$

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## Rainbow Connection Number in the Brick Product

Graphs  $C(2n, m, r)$

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**Abstract:** Let  $G$  be a nontrivial connected graph on which is defined a coloring  $c : E(G) \rightarrow \{1, 2, \dots, k\}$ ,  $k \in N$  of the edges of  $G$ , where adjacent edges may be colored the same. A path in  $G$  is called a rainbow path if no two edges of it are colored the same.  $G$  is rainbow connected if  $G$  contains a rainbow  $u - v$  path for every two vertices  $u$  and  $v$  in it. The minimum  $k$  for which there exists such a  $k$ -edge coloring is called the rainbow connection number of  $G$ , denoted by  $rc(G)$ . In this paper we determine  $rc(G)$  for the brick product  $C(2n, m, r)$  associated with the even cycle  $C_{2n}$  for  $m = 1$  and odd  $r \geq 5$  such that  $n = r + 1, r + 2$  and  $n \geq r + 3$ . We also discuss the critical property of the graphs with  $n = r + 1$  and  $n = r + 2$  with respect to rainbow coloring.

**Key Words:** Diameter, edge-coloring, rainbow path, Smarandacely  $H$ -rainbow connected, rainbow connected, rainbow connection number, rainbow critical graph, brick product.

**AMS(2010):** 05C15.

### §1. Introduction

All graphs considered in this paper are simple, finite and undirected. Let  $G$  be a nontrivial connected graph with an edge coloring  $c : E(G) \rightarrow \{1, 2, \dots, k\}$ ,  $k \in N$ , where adjacent edges may be colored the same. A path in  $G$  is called a *rainbow path* if no two edges of it are colored the same. An edge colored graph  $G$  is said to be *Smarandacely  $H$ -rainbow connected to a graph  $H$*  if for any two vertices in a subgraph  $G' \preceq G$  isomorphic to  $H$ , there is always a rainbow path in  $G$  connecting them. Particularly, an edge colored graph  $G$  is said to be *rainbow connected* if for any two vertices in  $G$ , there is a rainbow path in  $G$  connecting them, i.e., Smarandacely  $G$ -rainbow connected. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge coloring that makes it rainbow connected, i.e., a coloring such that each edge has a distinct color. The minimum  $k$  for which there exist a rainbow  $k$ -coloring of  $G$  is called the *rainbow connection number* of  $G$ , denoted by  $rc(G)$ .

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<sup>1</sup>Received July 14, 2016, Accepted May 22, 2017.

Let  $c$  be a rainbow coloring of  $G$ . For any two vertices  $u$  and  $v$  of  $G$  a *rainbow  $u-v$  geodesic* in  $G$  is a rainbow  $u-v$  path of length  $d(u, v)$  where  $d(u, v)$  is the distance between  $u$  and  $v$ .  $G$  is termed *strongly rainbow connected* if  $G$  contains a rainbow  $u-v$  geodesic for every two vertices  $u$  and  $v$  in  $G$  and in this case the coloring  $c$  is called a *strong rainbow coloring* of  $G$ . The minimum  $k$  for which there exists a coloring  $c : E(G) \rightarrow \{1, 2, \dots, k\}$ ,  $k \in N$  of the edges of  $G$  such that  $G$  is strongly rainbow connected is called the *strong rainbow connection number* of  $G$ , denoted by  $src(G)$ . Thus  $rc(G) \leq src(G)$  for every connected graph  $G$ .

The rainbow connection number and the strong rainbow connection number are defined for every connected graph  $G$ , since every coloring that assigns distinct colors to the edges of  $G$  is both a rainbow coloring and a strong rainbow coloring and  $G$  is rainbow connected and strongly rainbow connected with respect to some coloring of the edges of  $G$ .

The concept of rainbow connectivity and strong rainbow connectivity were first introduced by Chartrand et al. [2] in 2008 as a means of strengthening the connectivity. In [2], the authors computed the rainbow connection numbers of several graph classes (complete graphs, trees, cycles, wheels and complete bipartite graphs). In [5] and [6] K.Srinivasa Rao and R.Murali, determined  $rc(G)$  and  $src(G)$  of the stacked book graph, the grid graph, the prism graph etc. They also discussed the critical property of these graphs with respect to rainbow coloring. In [7], the authors determined  $rc(G)$  of brick product graphs associated with even cycles and also discussed the critical property of these graphs with respect to rainbow coloring. An overview about rainbow connection number can be found in a book of Li and Sun in [4] and a survey by Li et.al. in [3].

## §2. Definitions

**Definition 2.1** A graph  $G$  is said to be *rainbow critical* if the removal of any edge from  $G$  increases the rainbow connection number of  $G$ , i.e. if  $rc(G) = k$  for some positive integer  $k$ , then  $rc(G - e) > k$  for any edge  $e$  in  $G$ .

The brick product of even cycles was introduced in a paper by B.Alspach et.al. [1] in which the Hamiltonian laceability properties of brick products was explored.

**Definition 2.2** (Brick product of even cycle) Let  $m$ ,  $n$  and  $r$  be positive integers. Let  $C_{2n} = v_0, v_1, v_2, \dots, v_{(2n-1)}$ ,  $v_0$  denote a cycle of order  $2n$ . The  $(m, r)$ -brick product of  $C_{2n}$  denoted by  $C(2n, m, r)$  is defined as follows:

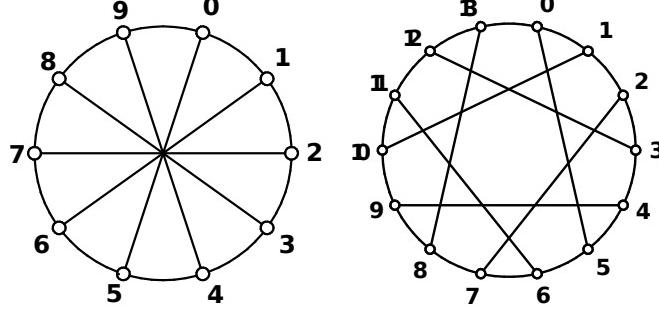
For  $m = 1$ , we require that  $r$  be odd and greater than 1. Then,  $C(2n, m, r)$  is obtained from  $C_{2n}$  by adding chords  $v_{2k}(v_{2k+r})$ ,  $k = 1, 2, \dots, n$ , where the computation is performed under modulo  $2n$ .

For  $m > 1$ , we require that  $m + r$  be even. Then,  $C(2n, m, r)$  is obtained by first taking disjoint union of  $m$  copies of  $C_{2n}$  namely,  $C_{2n}(1), C_{2n}(2), C_{2n}(3), \dots, C_{2n}(m)$  where for each  $i = 1, 2, \dots, m$ ,  $C_{2n}(i) = v_{i1}, v_{i2}, v_{i3}, \dots, v_{i(2n)}$ . Next, for each odd  $i = 1, 2, \dots, m-1$  and each even  $k = 0, 1, \dots, 2n-2$ , an edge(called a brick edge) drawn to join  $v_{ik}$  to  $v_{(i+1)k}$ , whereas, for each even  $i = 1, 2, \dots, m-1$  and each odd  $k = 1, 2, \dots, 2n-1$ , an edge (also called a brick

edge) is drawn to join  $v_{ik}$  to  $v_{(i+1)k}$ .

Finally, for each odd  $k = 1, 2, \dots, 2n - 1$ , an edge (called a hooking edge) is drawn to join  $v_{1k}$  to  $v_{m(k+r)}$ . An edge in  $C(2n, m, r)$  which is neither a brick edge nor a hooking edge is called a flat edge.

The brick products  $C(10, 1, 5)$  and  $C(14, 1, 5)$  are shown in the Figure 1.



**Figure 1** The brick product  $C(10, 1, 5)$  and  $C(14, 1, 5)$ .

In the next section, we determine the values of  $rc(G)$  for the brick product graph  $C(2n, m, r)$  for  $m = 1$  and  $n = r + 1$ ,  $n = r + 2$  and  $n \geq r + 3$  for odd  $r \geq 5$ . In our results, we denote the vertices of the cycle  $C_{2n}$  as  $v_0, v_1, \dots, v_{2n-1}, v_{2n} = v_0$ .

### §3. Main Results

**Theorem 3.1** Let  $G = C(2n, m, r)$ . Then for  $m = 1$  and odd  $r \geq 5$ ,

$$rc(G) = \begin{cases} \frac{n}{2} + 1 & \text{for } n = r + 1 \\ \lceil \frac{n}{2} \rceil & \text{for } n = r + 2 \end{cases}$$

*Proof* We consider the vertex set of  $G$  as  $V(G) = \{v_0, v_1, \dots, v_{2n-1}, v_{2n} = v_0\}$  and the edge set of  $G$  as  $E(G) = \{e_i : 1 \leq i \leq 2n\} \cup \{e'_i : 1 \leq i \leq n\}$ , where  $e_i$  is the *cycle edge*  $(v_{i-1}, v_i)$  and  $e'_i$  is the *brick edge*  $(v_{2k}, v_{2k+r})$ ,  $k = 0, 1, \dots, n$ . Here  $2k + r$  is computed modulo  $2n$ . We prove this result in different cases as follows.

**Case 1.**  $n = r + 1$ .

Since  $diam(G) = \frac{n}{2}$ , it follows that  $rc(G) \geq \frac{n}{2}$ . We define a  $\frac{n}{2}$  coloring  $c : E(G) \rightarrow \{1, 2, \dots, \frac{n}{2}\}$  on the cycle edges of  $G$  as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if } \frac{3n}{2} + 1 \leq i \leq 2n \end{cases}$$

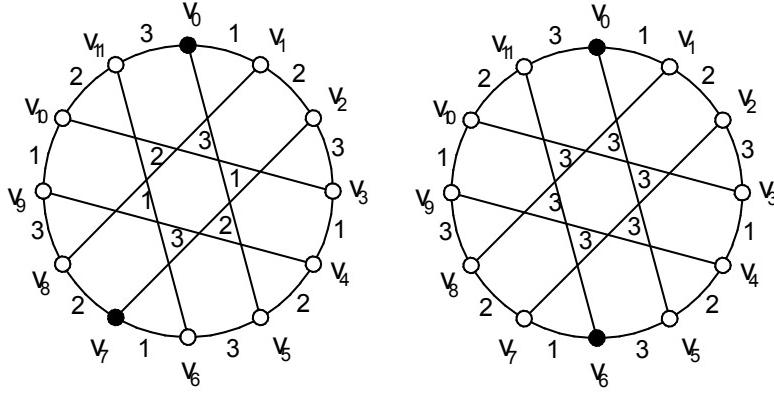
Also, we consider the assignment of colors to the brick edges in two different ways.

$$1. \quad c(e'_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

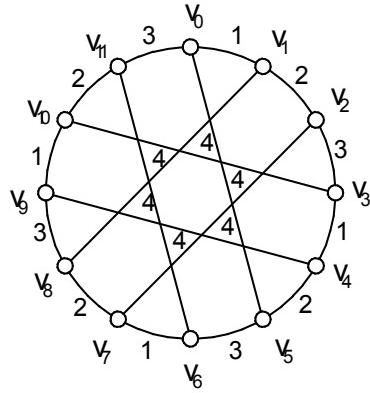
This coloring will not give a rainbow  $v_0 - v_{n+1}, v_1 - v_n$  path  $\forall n$  and this is true for other pair of vertices.

$$2. \quad c(e'_i) = \frac{n}{2} \quad \text{if } 1 \leq i \leq n.$$

This coloring also will not give a rainbow  $v_0 - v_n$  path  $\forall n$  and this is true for other pair of vertices. (This is illustrated in Figure 2).



**Figure 2** Assignment of colors in  $C(12, 1, 5)$  in two different ways.



**Figure 3** Assignment of colors in  $C(12, 1, 5)$ .

Accordingly, we define a coloring  $c : E(G) \rightarrow \{1, 2, \dots, \frac{n}{2} + 1\}$  and assign the colors to the edges of  $G$  as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if } \frac{3n}{2} + 1 \leq i \leq 2n \end{cases}$$

and

$$c(e'_i) = \frac{n}{2} + 1 \quad \text{if } 1 \leq i \leq n$$

From the above assignment, it is clear that for every two distinct vertices  $x, y \in V(G)$ , there exists an  $x - y$  rainbow path. Hence  $rc(G) \leq \frac{n}{2} + 1$ . This proves  $rc(G) = \frac{n}{2} + 1$ . An illustration for the assignment of colors in  $C(12, 1, 5)$  is provided in Figure 3.

**Case 2.**  $n = r + 2$ .

Since  $diam(G) = \lceil \frac{n}{2} \rceil$ , it follows that  $rc(G) \geq \lceil \frac{n}{2} \rceil$ . In order to show that  $rc(G) \leq \lceil \frac{n}{2} \rceil$ , we construct an edge coloring  $c : E(G) \rightarrow \{1, 2, \dots, \lceil \frac{n}{2} \rceil\}$  and assign the colors to the edges of  $G$  as

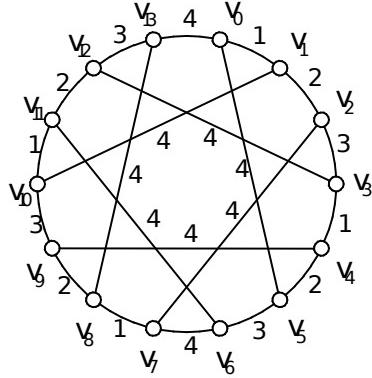
$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ i - \lfloor \frac{n}{2} \rfloor & \text{if } \lceil \frac{n}{2} \rceil \leq i \leq n - 1 \\ i - n & \text{if } n + 1 \leq i \leq \lfloor \frac{3n}{2} \rfloor \\ i - \lfloor \frac{3n}{2} \rfloor & \text{if } \lceil \frac{3n}{2} \rceil \leq i \leq 2n - 1 \\ \lceil \frac{n}{2} \rceil & \text{elsewhere} \end{cases}$$

and

$$c(e'_i) = \lceil \frac{n}{2} \rceil \quad \text{if } 1 \leq i \leq n$$

From the above assignment, for any two vertices  $x, y \in V(G)$ , we obtain a rainbow  $x - y$  path in  $G$ . Hence  $rc(G) \leq \lceil \frac{n}{2} \rceil$ .

This proves  $rc(G) = \lceil \frac{n}{2} \rceil$ . An illustration for the assignment of colors in  $C(14, 1, 5)$  is provided in Figure 4.  $\square$



**Figure 4** Assignment of colors in  $C(14, 1, 5)$ .

The critical nature of the brick product graph in Theorem 3.1 has been observed for particular values of  $n$ . This is illustrated in our next result.

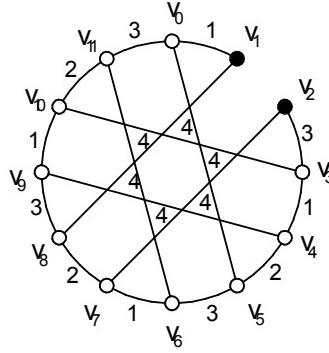
**Lemma 3.2** *Let  $G = C(2n, m, r)$ , where  $m = 1$  and odd  $r \geq 5$ . Then  $G$  is rainbow critical for  $n = r + 1$  and  $n = r + 2$ .*

$$\text{i.e., } rc(G - e) = n \text{ for } n = r + 1 \text{ and } n = r + 2.$$

*Proof* We prove this result in two cases.

**Case 1.**  $n = r + 1$ .

From the Theorem 3.1,  $rc(G) = \frac{n}{2} + 1$  for  $n = r + 1$ . Consider the graph  $G$  and let  $e \in E(G)$  be the any edge in  $G$ . Deletion of the edge  $e$  in  $G$  does not yield a rainbow path between the end vertices of  $e$  in  $G - e$ . For illustration see Figure 5.



**Figure 5** Assignment of colors in  $C(12, 1, 5)$ .

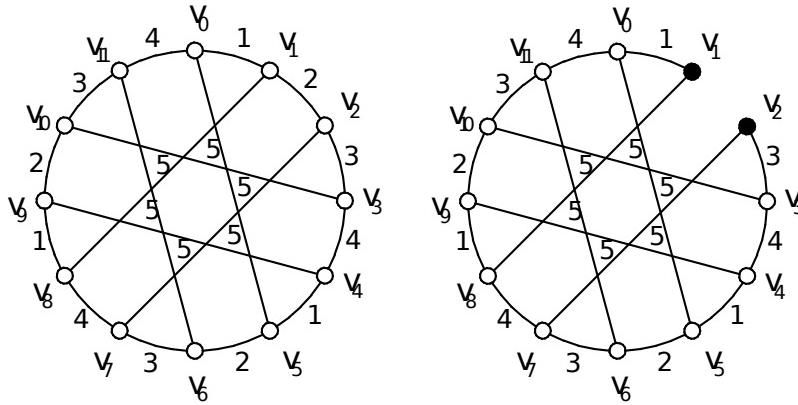
This shows  $rc(G - e) > \frac{n}{2} + 1$ . We define a rainbow coloring  $c : E(G) \rightarrow \{1, 2, \dots, \frac{n}{2} + 2\}$  to the edges of  $G$  as defined in Theorem 3.1. i.e.,

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} + 1 \\ i - (\frac{n}{2} + 1) & \text{if } \frac{n}{2} + 2 \leq i \leq n + 2 \\ i - (n + 2) & \text{if } n + 3 \leq i \leq \frac{3n}{2} + 3 \\ i - (\frac{3n}{2} + 3) & \text{if } \frac{3n}{2} + 4 \leq i \leq 2n \end{cases}$$

and

$$c(e'_i) = \frac{n}{2} + 2 \text{ if } 1 \leq i \leq n$$

From the above assignment of colors, we fail to get a rainbow path between any two arbitrary vertices in  $G - e$ . Without loss of generality let  $e = v_1v_2$ . An illustration is provided in Figure 6.



**Figure 6** Assignment of colors in  $C(12, 1, 5)$ .

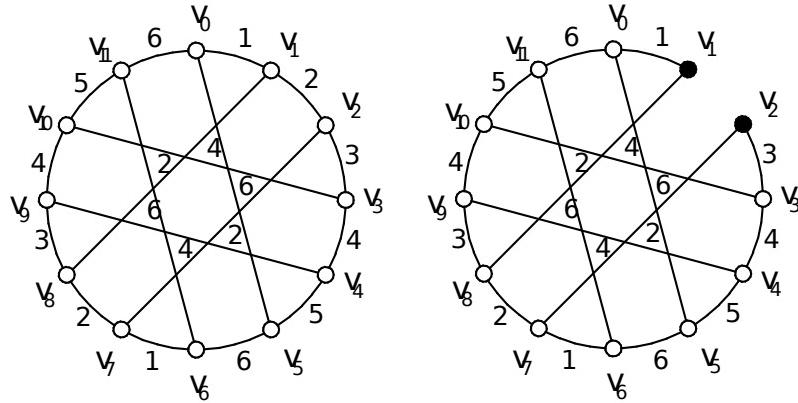
Accordingly, we assign a  $(\frac{n}{2} + \frac{r+1}{2}) = (\frac{n}{2} + \frac{n}{2}) = n$  rainbow coloring  $c : E(G) \rightarrow \{1, 2, \dots, n\}$  to the edges of  $G$  and since the cycle  $C_{2n}$  is a sub graph of  $G$ , assign the colors to the edges of cycle as in Theorem 3.1. That is,

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq 2n \end{cases}$$

Next, we will consider the assignment of colors to the brick edge as

$$c(e'_i) = \begin{cases} (2i + 4) \bmod n & \text{if } 1 \leq i \leq \frac{n}{2} \\ [(2i + 4) \bmod n] - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

From the above assignment, it is clear that for any two vertices  $x, y \in V(G)$ , there exists a rainbow  $x - y$  path in  $G - e$  and hence  $rc(G - e) \leq n$ . This proves  $rc(G - e) = n$ . An illustration for the assignment of colors in  $C(12, 1, 5)$  is provided in Figure 7.



**Figure 7** Assignment of colors in  $C(12, 1, 5)$ .

**Case 2.**  $n = r + 2$ .

The proof for  $n = r + 2$  is the same as Case 1.

As in Case 1, in this case also we fail to get the  $x - y$  rainbow path for different combinations of assignment of  $n - 1$  colors to the edges of  $G - e$ .

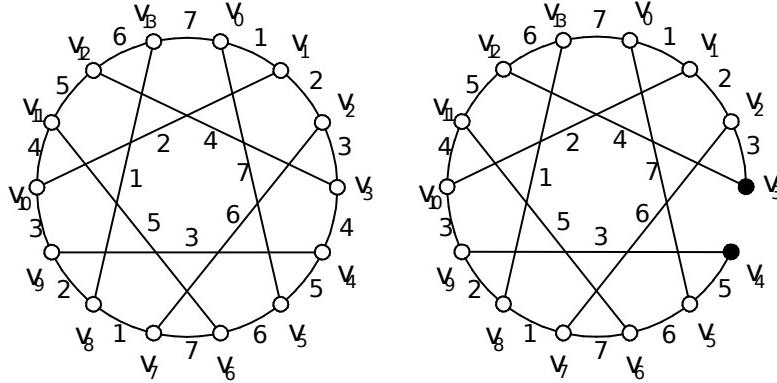
Accordingly, we construct a rainbow coloring  $c : E(G) \rightarrow \{1, 2, \dots, n\}$  to the edges of  $G$  as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq 2n \end{cases}$$

and

$$c(e'_i) = \begin{cases} n + 1 - i & \text{if } 1 \leq i \leq 2 \\ i & \text{if } i = 3 \\ i + 1 & \text{if } 4 \leq i \leq n - 3 \\ i - (n - 3) & \text{if } n - 2 \leq i \leq n - 1 \\ 4 & \text{if } i = n \end{cases}$$

From the above assignment for any two vertices  $x, y \in V(G)$ , we obtain a rainbow  $x - y$  path in  $G - e$ . This proves  $rc(G - e) = n$ . Hence the proof. An illustration for the assignment of colors in  $C(14, 1, 5)$  is provided in Figure 8.



**Figure 8** Assignment of colors in  $C(14, 1, 5)$ .

For  $r \geq 5$  and  $n \geq r + 3$ , we have the following result.

**Theorem 3.3** Let  $G = C(2n, m, r)$ . Then for  $m = 1$ , odd  $r \geq 5$  and  $n \geq r + 3$ ,

$$rc(G) = \begin{cases} \frac{n}{2} + 1 & \text{for } n = r + 3 \\ \lceil \frac{n}{2} \rceil + 1 & \text{for } r + 4 \leq n \leq 2r \\ (n - r + 1) - x \lfloor \frac{r}{2} \rfloor & \text{for } (2r + 1) + xr \leq n \leq 3r + xr \\ (2r + 2) + x(\lfloor \frac{r}{2} \rfloor + 1) & \text{for } (3r + 1) + xr \leq n \leq 4r + xr \\ & \text{where } x = 0, 2, 4 \dots \end{cases}$$

*Proof* We consider the vertex set  $V(G)$  and the edge set  $E(G)$  defined in Theorem 3.1. We prove this result in different cases as follows.

**Case 1.**  $n = r + 3$ .

We have two subcases.

**Subcase 1.1**  $n = 4k$ , where  $k = 2, 3, \dots$ .

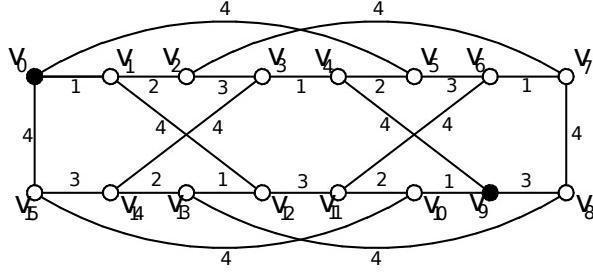
Here  $diam(G) = \frac{n+8}{4}$ . Clearly  $rc(G) \geq diam(G)$ . If we assign  $diam(G)$ -colors to the edges of  $G$ , we fail to obtain a rainbow path between a pair of vertices in  $G$ . This holds up to  $\frac{n}{2}$  colors. Hence, to prove this we consider an edge coloring  $c : E(G) \rightarrow \{1, 2, \dots, \frac{n}{2}\}$  to the edges of  $G$  as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if } \frac{3n}{2} + 1 \leq i \leq 2n \end{cases}$$

and

$$c(e'_i) = \frac{n}{2} \quad \text{if } 1 \leq i \leq n$$

This coloring will not yield a rainbow  $v_0 - v_{n+1}$  path  $\forall n$ , which is illustrated in Figure 9.



**Figure 9** Assignment of colors in  $C(16, 1, 5)$ .

**Subcase 1.2**  $n = 5k$ , where  $k = 2, 3, \dots$ .

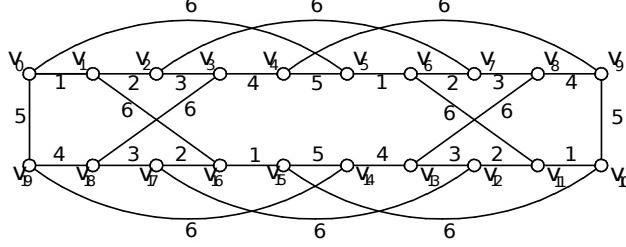
Here  $diam(G) = \frac{n+6}{4}$ . Clearly  $rc(G) \geq diam(G)$ . Similar to subcase 1 we fail to obtain a rainbow path between a pair of vertices in  $G$  up to  $\frac{n}{2}$ -colors. From Subcase 1 and Subcase 2, we have  $rc(G) \geq \frac{n}{2} + 1$ . Accordingly, we construct an edge coloring  $c : E(G) \rightarrow \{1, 2, \dots, \frac{n}{2} + 1\}$  and assign the colors to the edges of  $G$  as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if } \frac{3n}{2} + 1 \leq i \leq 2n \end{cases}$$

and

$$c(e'_i) = \frac{n}{2} + 1 \quad \text{if } 1 \leq i \leq n$$

From the above assignment, it is clear that for any two vertices  $x, y \in V(G)$ , there exists a rainbow  $x - y$  path in  $G$  and hence  $rc(G) \leq \frac{n}{2} + 1$ . This proves  $rc(G) = \frac{n}{2} + 1$ . An illustration for the assignment of colors in  $C(20, 1, 5)$  is provided in Figure 10.



**Figure 10** Assignment of colors in  $C(20, 1, 5)$ .

**Case 2.**  $r + 4 \leq n \leq 2r$ .

We consider the following subcases.

**Subcase 2.1**  $n$  is even.

The proof is similar to the proof provided in Subcase 1.1 of Case 1.

If we consider assignment of  $\frac{n}{2}$  colors in any combination to the edges of  $G$ , we fail to obtain a rainbow  $v_0 - v_{n+1}$  path  $\forall n$  (as illustrated in Figure 9).

This shows  $rc(G) \geq \frac{n}{2} + 1$ . It remains to show that  $rc(G) \leq \frac{n}{2} + 1$ . Define a coloring  $c : E(G) \rightarrow \{1, 2, \dots, \frac{n}{2} + 1\}$  and consider the assignment of colors to the edges of  $G$  as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if } \frac{3n}{2} + 1 \leq i \leq 2n \end{cases}$$

and

$$c(e'_i) = \frac{n}{2} + 1 \quad \text{if } 1 \leq i \leq n$$

From the above assignment, it is clear that for any two vertices  $x, y \in V(G)$ , there exists a rainbow  $x - y$  path with coloring  $c$ . Hence  $rc(G) \leq \frac{n}{2} + 1$ . An illustration for the assignment of colors in  $C(20, 1, 5)$  is provided in Figure 10.

**Subcase 2.2**  $n$  is odd.

If we assign  $diam(G)$  colors to the edges of  $G$ , we fail to obtain a rainbow path between a pair of vertices in  $G$ . This holds up to  $\lceil \frac{n}{2} \rceil$  colors. Hence, to prove this we consider an edge

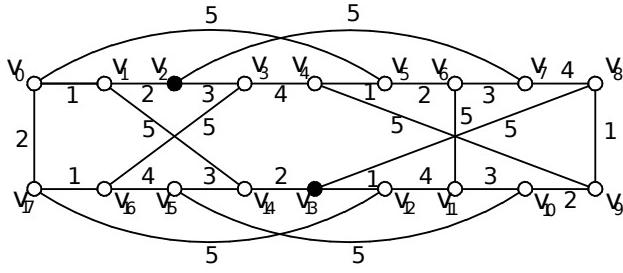
coloring  $c : E(G) \rightarrow \{1, 2, \dots, \lceil \frac{n}{2} \rceil\}$  to the edges of  $G$  as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ i - \lfloor \frac{n}{2} \rfloor & \text{if } \lceil \frac{n}{2} \rceil \leq i \leq n-1 \\ i - (n-1) & \text{if } n \leq i \leq \lfloor \frac{3n}{2} \rfloor - 1 \\ i - (\lfloor \frac{3n}{2} \rfloor - 1) & \text{if } \lceil \frac{3n}{2} \rceil \leq i \leq 2n-2 \\ i - (2n-2) & \text{if } 2n-1 \leq i \leq 2n \end{cases}$$

and

$$c(e'_i) = \lceil \frac{n}{2} \rceil \text{ if } 1 \leq i \leq n$$

This coloring will not yield a rainbow  $v_2 - v_{\frac{3n-1}{2}}$  path  $\forall n$ , which is illustrated in Figure 11.



**Figure 11** Assignment of colors in  $C(18, 1, 5)$ .

This shows  $rc(G) \geq \lceil \frac{n}{2} \rceil + 1$ . It remains to show that  $rc(G) \leq \lceil \frac{n}{2} \rceil + 1$ . We construct an edge coloring  $c : E(G) \rightarrow \{1, 2, \dots, \lceil \frac{n}{2} \rceil + 1\}$  and assign the colors to the edges of  $G$  as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ i - \lfloor \frac{n}{2} \rfloor & \text{if } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n-1 \\ \lceil \frac{n}{2} \rceil & \text{if } i = n \text{ and } i = 2n \\ i - n & \text{if } n+1 \leq i \leq \lfloor \frac{3n}{2} \rfloor \\ i - \lfloor \frac{3n}{2} \rfloor & \text{if } \lceil \frac{3n}{2} \rceil + 1 \leq i \leq 2n-1 \end{cases}$$

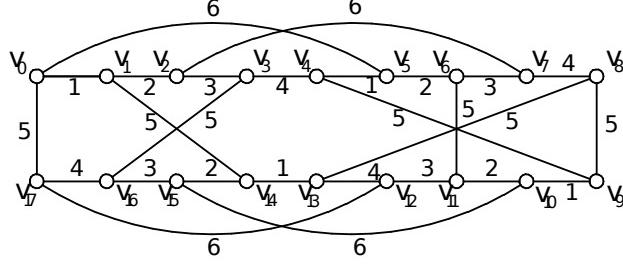
and

$$c(e'_i) = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if } 1 \leq i \leq \frac{n-r}{2} \text{ and } \frac{n+3}{2} \leq i \leq \frac{2n-r+1}{2} \\ \lceil \frac{n}{2} \rceil & \text{if } \frac{n-r+2}{2} \leq i \leq \frac{n+1}{2} \text{ and } \frac{2n-r+3}{2} \leq i \leq n \end{cases}$$

From the above assignment, it is clear that for any two vertices  $x, y \in V(G)$ , there exists a rainbow  $x - y$  path. Hence  $rc(G) \leq \lceil \frac{n}{2} \rceil + 1$ .

Combining both the Subcases, we have  $rc(G) \leq \lceil \frac{n}{2} \rceil + 1, \forall n$  such that  $r+4 \leq n \leq 2r$ .

This proves  $rc(G) = \lceil \frac{n}{2} \rceil + 1$ . An illustration for the assignment of colors in  $C(18, 1, 5)$  is provided in Figure 12.

**Figure 12** Assignment of colors in  $C(18, 1, 5)$ .

**Case 3.** For  $(2r+1)+xr \leq n \leq 3r+xr$  and  $(3r+1)+xr \leq n \leq 4r+xr$  where  $x = 0, 2, 4, \dots$

Let

$$h = \begin{cases} (n - r + 1) - x\lfloor \frac{r}{2} \rfloor & \text{for } (2r+1) + xr \leq n \leq 3r + xr \\ (2r+2) + x(\lfloor \frac{r}{2} \rfloor + 1) & \text{for } (3r+1) + xr \leq n \leq 4r + xr \\ & \text{where } x = 0, 2, 4, \dots \end{cases}$$

As in Case 1 and Case 2 we again fail to obtain a rainbow path between a pair of vertices in  $G$  up to  $h - 1$  colors. This shows  $rc(G) \geq h$ .

To show that  $rc(G) \leq h$ , we construct a rainbow coloring  $c : E(G) \rightarrow \{1, 2, \dots, (n - r + 1) - x\lfloor \frac{r}{2} \rfloor\}$  for  $(2r+1) + xr \leq n \leq 3r + xr$  and  $c : E(G) \rightarrow \{1, 2, \dots, (2r+2) + x(\lfloor \frac{r}{2} \rfloor + 1)\}$  for  $(3r+1) + xr \leq n \leq 4r + xr$  and assign the colors to the edges of  $G$ . Before assignment of colors, we split  $E(G)$  as,  $e_i$  where  $1 \leq i \leq n - 1$ ,  $i = n$ ,  $i = 2n$  and  $n + 1 \leq i \leq 2n - 1$  (cycle edges) and  $e'_i$  where  $1 \leq i \leq n$  (brick edges). we assign the colors to the cycle edges from  $e_1$  to  $e_{n-1}$  as

$$c(e_i) = \begin{cases} i & \text{for } 1 \leq i \leq r \\ i - kr & \text{for } (2k-1)r + 1 \leq i \leq l \end{cases}$$

where

$$l = \begin{cases} (2k+1)r & \text{for each } k = 1, 2, \dots, \lceil \frac{n-r}{2r} \rceil - 1 \\ n-1 & \text{for } k = \lceil \frac{n-r}{2r} \rceil \end{cases}$$

and assign the colors to the cycle edge  $e_n$  and  $e_{2n}$  as

$$c(e_n) = c(e_{2n}) = \begin{cases} [(n-r) - x\lfloor \frac{r}{2} \rfloor] - \frac{x}{2} & \text{for } (2r+1) + xr \leq n \leq 3r + xr \\ 2r + x(\lfloor \frac{r}{2} \rfloor + 1) - \frac{x}{2} & \text{for } (3r+1) + xr \leq n \leq 4r + xr \\ & \text{where } x = 0, 2, \dots \end{cases}$$

Also, we assign the colors to the cycle edges from  $e_{n+1}$  to  $e_{2n-1}$  as

$$c(e_i) = \begin{cases} i - n & \text{for } n+1 \leq i \leq n+r \\ i - (n+kr) & \text{for } n+(2k-1)r+1 \leq i \leq m \end{cases}$$

where

$$m = \begin{cases} n + (2k+1)r & \text{for each } k = 1, 2, \dots, \lceil \frac{n-r}{2r} \rceil - 1 \\ 2n - 1 & \text{for } k = \lceil \frac{n-r}{2r} \rceil \end{cases}$$

and assign the colors to the brick edges from  $e'_1$  to  $e'_n$  as

$$c(e_i') = \begin{cases} c(e_n) + k & \text{for } (k-1)r+1 \leq i \leq l \\ c(e_n) & \text{for } \lfloor \frac{n-4}{2} \rfloor + 1 \leq i \leq \lceil \frac{n}{2} \rceil \\ c(e_n) + k & \text{for } \lceil \frac{n}{2} \rceil + (k-1)r+1 \leq i \leq m \\ c(e_n) & \text{for } (n-1) \leq i \leq n \end{cases}$$

where

$$l = \begin{cases} kr & \text{for each } k = 1, 2, \dots, \lceil \frac{n-r}{2r} \rceil - 1 \\ \lfloor \frac{n-4}{2} \rfloor & \text{for } k = \lceil \frac{n-r}{2r} \rceil \end{cases}$$

and

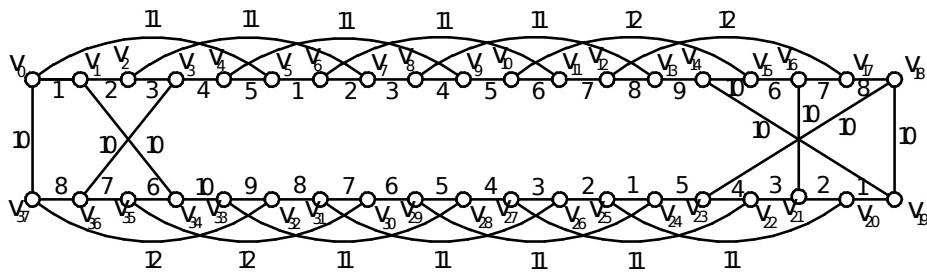
$$m = \begin{cases} \lceil \frac{n}{2} \rceil + kr & \text{for each } k = 1, 2, \dots, \lceil \frac{n-r}{2r} \rceil - 1 \\ n - 2 & \text{for } k = \lceil \frac{n-r}{2r} \rceil \end{cases}$$

From the above assignment, it is clear that for any two vertices  $x, y \in V(G)$ , there exists a rainbow  $x - y$  path and hence

$$rc(G) = \begin{cases} (n - r + 1) - x\lfloor \frac{r}{2} \rfloor & \text{for } (2r + 1) + xr \leq n \leq 3r + xr \\ (2r + 2) + x(\lfloor \frac{r}{2} \rfloor + 1) & \text{for } (3r + 1) + xr \leq n \leq 4r + xr \end{cases}$$

where  $x = 0, 2, 4, \dots$

Hence the proof. An illustration for the assignment of colors in  $C(38, 1, 5)$  is provided in Figure 13.  $\square$



**Figure 13** Assignment of colors in  $C(38, 1, 5)$ .

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## Mannheim Partner Curve a Different View

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**Abstract:** In this study, we investigated special Smarandache curves belonging to Sabban frame drawn on the surface of the sphere by Darboux vector of Mannheim partner curve . We created Sabban frame belonging to this curve. It were explained Smarandache curves position vector is consisted by Sabban vectors belonging to this curve. Then, we calculated geodesic curvatures of this Smarandache curves. Found results were expressed depending on the Mannheim curve.

**Key Words:** Mannheim curve pair, Darboux vector, Smarandache curves, Sabban frame, geodesic curvature.

**AMS(2010):** 53A04.

### §1. Introduction and Preliminaries

Let  $\alpha : I \rightarrow E^3$  be a unit speed curve denote by  $\{T, N, B, p, q\}$  the moving Frenet apparatus. The Frenet formulae is given by ([5])

$$\begin{cases} T'(s) = p(s)N(s) \\ N'(s) = -p(s)T(s) + q(s)B(s) \\ B'(s) = -q(s)N(s). \end{cases} \quad (1.1)$$

The Darboux vector defined by

$$W = qT + pB.$$

By the unit Darboux vector, we have

$$\sin \varphi = \frac{q}{\sqrt{p^2 + q^2}}, \quad \cos \varphi = \frac{p}{\sqrt{p^2 + q^2}}$$

including

$$C = \sin \varphi T + \cos \varphi B$$

where  $\angle(W, B) = \varphi$ , ([4]). Let  $\alpha$  and  $\alpha_1$  be the  $C^2$ -class differentiable two curves and  $T_1(s)$ ,  $N_1(s)$ ,  $B_1(s)$  be the Frenet vectors of  $\alpha_1$ . If the binormal vector of the curve  $\alpha_1$  is linearly dependent on the principal normal vector of the curve  $\alpha$ , then  $(\alpha)$  is defined a Mannheim curve

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<sup>1</sup>Received July 12, 2016, Accepted May 23, 2017.

and  $(\alpha_1)$  a Mannheim partner curve of  $(\alpha)$ . The relations between the Frenet vectors we can write [3, 6, 7]

$$\begin{cases} T_1 = \cos \sigma T - \sin \sigma B \\ N_1 = \sin \sigma T + \cos \sigma B \\ B_1 = N \end{cases} \quad (1.2)$$

and for the curvatures we get

$$\begin{cases} \bar{p} = \frac{p\sigma'}{\lambda q \sqrt{p^2 + q^2}} \\ \bar{q} = \frac{p}{\lambda q}. \end{cases} \quad (1.3)$$

Let  $(\alpha, \alpha_1)$  be a curve pair in  $\mathbb{E}^3$ . For the vector  $C_1$  is the direction of the Mannheim partner curve  $\alpha_1$  we have [3,8]

$$C_1 = \frac{\sqrt{p^2 + q^2}}{\sqrt{p^2 + q^2 + \sigma'^2}} C + \frac{\sigma'}{\sqrt{p^2 + q^2 + \sigma'^2}} N. \quad (1.4)$$

Let  $\omega : I \rightarrow S^2$  be a unit speed spherical curve. We can write ([10])

$$\omega(s) = \omega(s), \quad t(s) = \omega'(s), \quad d(s) = \omega(s) \wedge t(s), \quad (1.5)$$

where,  $\{\omega(s), t(s), d(s)\}$  frame is expressed the Sabban frame of  $\omega$  on  $S^2$ . Then we have equations ([10])

$$\omega'(s) = t(s), \quad t'(s) = -\omega(s) + \kappa_g(s)d(s), \quad d'(s) = -\kappa_g(s)t(s), \quad (1.6)$$

where  $\kappa_g$  is expressed the geodesic curvature of the curve  $\omega$  on  $S^2$  which is ([10])

$$\kappa_g(s) = \langle t'(s), d(s) \rangle. \quad (1.7)$$

## §2. Mannheim Partner Curve a Different View

Let  $(C_1)$  be a unit speed spherical curve on  $S^2$ . Then we can write

$$\begin{cases} C_1 = \sin \bar{\varphi} T_1 + \cos \bar{\varphi} B_1, \\ T_{C_1} = \cos \bar{\varphi} T_1 - \sin \bar{\varphi} B_1, \\ C_1 \wedge T_{C_1} = N_1, \end{cases} \quad (2.1)$$

where  $\angle(C_1, B_1) = \bar{\varphi}$ . Then from the equation (1.6) we have the following equations of  $(C_1)$  are

$$\begin{cases} C_1' = T_{C_1} \\ T_{C_1}' = -C_1 + \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} C_1 \wedge T_{C_1} \\ (C_1 \wedge T_{C_1})' = -\frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} T_{C_1}. \end{cases} \quad (2.2)$$

From the equation (1.7), we have the following geodesic curvature of  $(C_1)$  is

$$\kappa_g = \langle T'_{C_1}, C_1 \wedge T_{C_1} \rangle \implies \kappa_g = \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\varphi'}. \quad (2.3)$$

**2.1** The  $\beta_1$ -Smarandache curve can be defined by

$$\beta_1(s) = \frac{1}{\sqrt{2}}(C_1 + T_{C_1}) \quad (2.4)$$

or substituting the equation (2.1) into equation (2.4) we obtain

$$\beta_1(s) = \frac{1}{\sqrt{2}} \left( (\sin \bar{\varphi} + \cos \bar{\varphi}) T_1 + (\cos \bar{\varphi} - \sin \bar{\varphi}) B_1 \right). \quad (2.5)$$

Differentiating (2.4) we can write

$$T_{\beta_1} = \frac{\bar{\varphi}'(\cos \bar{\varphi} - \sin \bar{\varphi})}{\sqrt{2\bar{\varphi}'^2 + \bar{p}^2 + \bar{q}^2}} T_1 + \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\sqrt{2\bar{\varphi}'^2 + \bar{p}^2 + \bar{q}^2}} N_1 - \frac{\bar{\varphi}'(\cos \bar{\varphi} + \sin \bar{\varphi})}{\sqrt{2\bar{\varphi}'^2 + \bar{p}^2 + \bar{q}^2}} B_1. \quad (2.6)$$

Considering the equations (2.5) and (2.6) we get

$$\beta_1 \wedge T_{\beta_1} = \frac{\sqrt{\bar{p}^2 + \bar{q}^2}(\cos \bar{\varphi} + \sin \bar{\varphi})}{\sqrt{2\bar{p}^2 + 2\bar{q}^2 + 4\bar{\varphi}'^2}} T_1 - \frac{\bar{\varphi}'}{\sqrt{2\bar{p}^2 + 2\bar{q}^2 + 4\bar{\varphi}'^2}} N_1 + \frac{\sqrt{\bar{p}^2 + \bar{q}^2}(\cos \bar{\varphi} + \sin \bar{\varphi})}{\sqrt{2\bar{p}^2 + 2\bar{q}^2 + 4\bar{\varphi}'^2}} B_1. \quad (2.7)$$

Differentiating (2.6) where coefficients

$$\begin{aligned} \chi_1 &= -2 - \left( \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \right)^2 + \left( \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \right)' \left( \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \right) \\ \chi_2 &= -2 - 3 \left( \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \right)^2 - \left( \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \right)^4 - \left( \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \right)' \left( \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \right) \\ \chi_3 &= 2 \left( \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \right) + \left( \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \right)^3 + \left( \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \right)' \end{aligned} \quad (2.8)$$

including we can reach,

$$T'_{\beta_1} = \frac{\bar{\varphi}'^4 \sqrt{2}(\chi_1 \sin \bar{\varphi} + \chi_2 \cos \bar{\varphi})}{(\bar{p}^2 + \bar{q}^2 + \bar{\varphi}'^2)^2} T_1 + \frac{\chi_3 \bar{\varphi}'^4 \sqrt{2}}{(\bar{p}^2 + \bar{q}^2 + \bar{\varphi}'^2)^2} N_1 + \frac{\bar{\varphi}'^4 \sqrt{2}(\chi_1 \cos \bar{\varphi} - \chi_2 \sin \bar{\varphi})}{(\bar{p}^2 + \bar{q}^2 + \bar{\varphi}'^2)^2} B_1. \quad (2.9)$$

From the equation (2.7) and (2.9)  $\kappa_g^{\beta_1}$  geodesic curvature for Mannheim partner curve  $\beta_1$  is

$$\begin{aligned} \kappa_g^{\beta_1} &= \langle T'_{\beta_1}, (C_1 \wedge T_{C_1})_{\beta_1} \rangle \\ &= \frac{1}{\left( 2 + \left( \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \right)^2 \right)^{\frac{5}{2}}} \left( \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \chi_1 - \frac{\sqrt{\bar{p}^2 + \bar{q}^2}}{\bar{\varphi}'} \chi_2 + 2\chi_3 \right). \end{aligned} \quad (2.10)$$

From the equation (1.2) and (1.3) Sabban apparatus of the  $\beta_1$ -Smarandache curve for

Mannheim curve are

$$\begin{aligned}
\beta_1(s) &= \frac{(\sigma' + \sqrt{p^2 + q^2}) \cos \sigma}{\sqrt{2\sigma'^2 + 2p^2 + q^2}} T - \frac{\sigma' - \sqrt{p^2 + q^2}}{\sqrt{2\sigma'^2 + 2p^2 + q^2}} N - \frac{(\sigma' + \sqrt{p^2 + q^2}) \sin \sigma}{\sqrt{2\sigma'^2 + 2p^2 + q^2}} B, \\
T_{\beta_1} &= \frac{(\sigma' - \sqrt{p^2 + q^2}) \eta \cos \sigma - \sqrt{p^2 + q^2 + \sigma'^2} \cos \sigma}{\sqrt{p^2 + q^2 + \sigma'^2} \sqrt{1 + 2\eta^2}} T + \frac{\eta(\sigma' + \sqrt{p^2 + q^2})}{\sqrt{p^2 + q^2 + \sigma'^2} \sqrt{1 + 2\eta^2}} N \\
&\quad + \frac{(\sqrt{p^2 + q^2} - \sigma') \eta \sin \sigma + \sqrt{p^2 + q^2 + \sigma'^2} \cos \sigma}{\sqrt{p^2 + q^2 + \sigma'^2} \sqrt{1 + 2\eta^2}} B, \\
\beta_1 \wedge T_{\beta_1} &= \frac{(\sqrt{p^2 + q^2} - \sigma') \cos \sigma + 2\eta \sqrt{p^2 + q^2 + \sigma'^2} \sin \sigma}{\sqrt{2 + 4\eta^2} \sqrt{p^2 + q^2 + \sigma'^2}} T + \frac{\sigma' + \eta \sqrt{p^2 + q^2}}{\sqrt{2 + 4\eta^2} \sqrt{p^2 + q^2 + \sigma'^2}} N \\
&\quad - \frac{(\sqrt{p^2 + q^2} + \sigma') \sin \sigma - 2\eta \sqrt{p^2 + q^2 + \sigma'^2} \cos \sigma}{\sqrt{2 + 4\eta^2} \sqrt{p^2 + q^2 + \sigma'^2}} B, \\
T'_{\beta_1} &= \frac{(\bar{\chi}_1 \sqrt{p^2 + q^2} + \bar{\chi}_2 \sigma') \eta^4 \sqrt{2} \cos \sigma - \bar{\chi}_3 \eta^4 \sqrt{2p^2 + q^2 + 2\sigma'^2} \sin \sigma}{(1 + 2\eta^2)^2 \sqrt{p^2 + q^2 + \sigma'^2}} T \\
&\quad + \frac{(\bar{\chi}_1 \sigma' - \bar{\chi}_2 \sqrt{p^2 + q^2}) \eta^4 \sqrt{2}}{(1 + 2\eta^2)^2 \sqrt{p^2 + q^2 + \sigma'^2}} N \\
&\quad + \frac{(\bar{\chi}_2 \sigma' - \bar{\chi}_1 \sqrt{p^2 + q^2}) \eta^4 \sqrt{2} \sin \sigma + \bar{\chi}_3 \eta^4 \sqrt{2p^2 + q^2 + 2\sigma'^2} \cos \sigma}{(1 + 2\eta^2)^2 \sqrt{p^2 + q^2 + \sigma'^2}} B
\end{aligned}$$

and

$$\kappa_g^{\beta_1} = \frac{1}{(2 + \frac{1}{\eta^2})^{\frac{5}{2}}} \left( \frac{1}{\eta} \bar{\chi}_1 - \frac{1}{\eta} \bar{\chi}_2 + 2\bar{\chi}_3 \right), \quad (2.11)$$

where

$$\frac{1}{\eta} = \frac{(\bar{\varphi})'}{\sqrt{p^2 + q^2}} = \left( \frac{\sqrt{p^2 + q^2}}{\sqrt{\sigma'^2 + p^2 + q^2}} \right)' \frac{\lambda q \sqrt{p^2 + q^2}}{\sigma'} \quad (2.12)$$

and

$$\begin{cases} \bar{\chi}_1 = -2 - \frac{1}{\eta^2} + \frac{1}{\eta} \frac{1}{\eta} \\ \bar{\chi}_2 = -2 - 3 \frac{1}{\eta^2} - \frac{1}{\eta^4} - \frac{1}{\eta} \frac{1}{\eta} \\ \bar{\chi}_3 = 2 \frac{1}{\eta} + \frac{1}{\eta^3} + 2 \frac{1}{\eta} \end{cases} \quad (2.13)$$

**2.2** The  $\beta_2$ -Smarandache curve can be defined by

$$\beta_2(s) = \frac{1}{\sqrt{2}} (C_1 + C_1 \wedge T_{C_1}) \quad (2.14)$$

or from the equation (1.2), (1.4) and (2.1) we can write

$$\begin{aligned}\beta_2(s) = & \frac{\sqrt{p^2 + q^2} \cos \sigma + \sqrt{\sigma'^2 + p^2 + q^2} \sin \sigma}{\sqrt{2\sigma'^2 + 2p^2 + q^2}} T + \frac{\sigma'}{\sqrt{2\sigma'^2 + 2p^2 + q^2}} N \\ & + \frac{\sqrt{\sigma'^2 + p^2 + q^2} \cos \sigma - \sqrt{p^2 + q^2} \sin \sigma}{\sqrt{2\sigma'^2 + 2p^2 + q^2}} B.\end{aligned}\quad (2.15)$$

Differentiating (2.15) we can write

$$T_{\beta_2} = \frac{\sigma' \cos \sigma}{\sqrt{p^2 + q^2 + \sigma'^2}} T - \frac{\sqrt{p^2 + q^2}}{\sqrt{p^2 + q^2 + \sigma'^2}} N - \frac{\sigma' \sin \sigma}{\sqrt{p^2 + q^2 + \sigma'^2}} B. \quad (2.16)$$

Considering the equations (2.15) and (2.16) it is easily seen

$$\begin{aligned}\beta_2 \wedge T_{\beta_2} = & \frac{\sqrt{p^2 + q^2 + \sigma'^2} \sin \sigma - \sqrt{p^2 + q^2} \cos \sigma}{\sqrt{2p^2 + q^2 + 2\sigma'^2}} T - \frac{\sigma'}{\sqrt{2p^2 + q^2 + 2\sigma'^2}} N \\ & + \frac{\sqrt{p^2 + q^2} \sin \sigma + \sqrt{p^2 + q^2 + \sigma'^2} \cos \sigma}{\sqrt{2p^2 + q^2 + 2\sigma'^2}} B.\end{aligned}\quad (2.17)$$

Differentiating (2.16) we can write

$$\begin{aligned}T'_{\beta_2} = & \frac{\sqrt{2p^2 + q^2 + 2\sigma'^2} \sin \sigma - \eta \sqrt{2} \sqrt{p^2 + q^2} \cos \sigma}{(\eta - 1) \sqrt{p^2 + q^2 + \sigma'^2}} T + \frac{\eta \sqrt{2} \sigma'}{(\eta - 1) \sqrt{p^2 + q^2 + \sigma'^2}} N \\ & + \frac{\eta \sqrt{2} \sqrt{p^2 + q^2} \sin \sigma + \sqrt{2p^2 + q^2 + 2\sigma'^2} \cos \sigma}{(\eta - 1) \sqrt{p^2 + q^2 + \sigma'^2}} B,\end{aligned}\quad (2.18)$$

where,  $\kappa_g^{\beta_2}$  geodesic curvature for  $\beta_2$  is

$$\kappa_g^{\beta_2} = \frac{1 + \eta}{\eta - 1}. \quad (2.19)$$

**2.3** The  $\beta_3$ -Smarandache curves can be defined by

$$\beta_3(s) = \frac{1}{\sqrt{2}} (T_{C_1} + C_1 \wedge T_{C_1}) \quad (2.20)$$

or from the equation (2.1), (1.4) and (1.2) we can write

$$\begin{aligned}\beta_3(s) = & \frac{\sigma' \cos \sigma + \sqrt{p^2 + q^2 + \sigma'^2} \sin \sigma}{\sqrt{2\sigma'^2 + 2p^2 + q^2}} T + \frac{\sqrt{p^2 + q^2}}{\sqrt{2\sigma'^2 + 2p^2 + q^2}} N \\ & + \frac{\sqrt{p^2 + q^2 + \sigma'^2} \cos \sigma - \sigma' \sin \sigma}{\sqrt{2\sigma'^2 + 2p^2 + q^2}} B.\end{aligned}\quad (2.21)$$

Differentiating (2.21) we reach

$$\begin{aligned} T_{\beta_3} &= \frac{\sqrt{p^2 + q^2 + \sigma'^2} \sin \sigma - (\sigma' + \eta \sqrt{p^2 + q^2}) \cos \sigma}{\sqrt{2 + \eta^2} \sqrt{p^2 + q^2 + \sigma'^2}} T + \frac{\sqrt{p^2 + q^2} - \eta \sigma'}{\sqrt{2 + \eta^2} \sqrt{p^2 + q^2 + \sigma'^2}} N \\ &\quad + \frac{(\eta \sqrt{p^2 + q^2} + \sigma') \sin \sigma + \sqrt{p^2 + q^2 + \sigma'^2} \cos \sigma}{\sqrt{2 + \eta^2} \sqrt{p^2 + q^2 + \sigma'^2}} B. \end{aligned} \quad (2.22)$$

Considering the equations (2.21) and (2.22) it is easily seen

$$\begin{aligned} \beta_3 \wedge T_{\beta_3} &= \frac{(2\sqrt{p^2 + q^2} - \eta \sigma') \cos \sigma + \eta \sqrt{p^2 + q^2 + \sigma'^2} \sin \sigma}{\sqrt{4 + 2\eta^2} \sqrt{p^2 + q^2 + \sigma'^2}} T \\ &\quad + \frac{2\sigma' - \eta \sqrt{p^2 + q^2}}{\sqrt{4 + 2\eta^2} \sqrt{p^2 + q^2 + \sigma'^2}} N \\ &\quad + \frac{(\eta \sigma' - 2\sqrt{p^2 + q^2}) \sin \sigma + \eta \sqrt{p^2 + q^2 + \sigma'^2} \cos \sigma}{\sqrt{4 + 2\eta^2} \sqrt{p^2 + q^2 + \sigma'^2}} B. \end{aligned} \quad (2.23)$$

Differentiating (2.22) where

$$\begin{cases} \bar{\delta}_1 = \frac{1}{\eta} + 2\frac{1}{\eta^3} + 2\frac{1}{\eta'}\frac{1}{\eta} \\ \bar{\delta}_2 = -1 - 3\frac{1}{\eta^2} - 2\frac{1}{\eta^4} - \frac{1}{\eta'} \\ \bar{\delta}_3 = -\frac{1}{\eta^2} - 2\frac{1}{\eta^4} + \frac{1}{\eta'} \end{cases} \quad (2.24)$$

including we have

$$\begin{aligned} T'_{\beta_3} &= \frac{(\bar{\delta}_2 \sqrt{p^2 + q^2} - \bar{\delta}_1 \sigma') \eta^4 \sqrt{2} \cos \sigma + \bar{\delta}_3 \eta^4 \sqrt{2p^2 + q^2 + 2\sigma'^2} \sin \sigma}{(2 + \eta^2)^2 \sqrt{\sigma'^2 + p^2 + q^2}} T \\ &\quad + \frac{\eta^4 \sqrt{2} (\bar{\delta}_2 \sigma' + \bar{\delta}_1 \sqrt{p^2 + q^2})}{(2 + \eta^2)^2 \sqrt{\sigma'^2 + p^2 + q^2}} N \\ &\quad + \frac{(\bar{\delta}_1 \sigma' - \bar{\delta}_2 \sqrt{p^2 + q^2}) \eta^4 \sqrt{2} \sin \sigma + \bar{\delta}_3 \eta^4 \sqrt{2p^2 + q^2 + 2\sigma'^2} \cos \sigma}{(2 + \eta^2)^2 \sqrt{\sigma'^2 + p^2 + q^2}} B, \end{aligned} \quad (2.25)$$

where,  $\kappa_g^{\beta_3}$  geodesic curvature for Mannheim curve  $\beta_3$  is

$$\kappa_g^{\beta_3} = \frac{1}{(2 + \eta^2)^{\frac{5}{2}}} (2\eta^5 \bar{\delta}_1 - \eta^4 \bar{\delta}_2 + \eta^4 \bar{\delta}_3). \quad (2.26)$$

**2.4** The  $\beta_4$ -Smarandache curves can be defined by

$$\beta_4(s) = \frac{1}{\sqrt{3}} (C_1 + T_{C_1} + C_1 \wedge T_{C_1}) \quad (2.27)$$

or from the equation (1.2), (1.4) and (2.1) we can write

$$\begin{aligned}\beta_4(s) &= \frac{(\sigma' + \sqrt{p^2 + q^2}) \cos \sigma + \sqrt{\sigma'^2 + p^2 + q^2} \sin \sigma}{\sqrt{3\sigma'^2 + 3p^2 + q^2}} T + \frac{\sigma' - \sqrt{p^2 + q^2}}{\sqrt{3\sigma'^2 + 3p^2 + q^2}} N \\ &\quad + \frac{\sqrt{\sigma'^2 + p^2 + q^2} \cos \sigma - (\sigma' + \sqrt{p^2 + q^2}) \sin \sigma}{\sqrt{3\sigma'^2 + 3p^2 + q^2}} B.\end{aligned}\quad (2.28)$$

Differentiating (2.28), we reach

$$\begin{aligned}T_{\beta_4} &= \frac{((\eta - 1)\sigma' - \eta\sqrt{p^2 + q^2}) \cos \sigma + \sqrt{p^2 + q^2 + \sigma'^2} \sin \sigma}{\sqrt{2(1 - \eta + \eta^2)} \sqrt{p^2 + q^2 + \sigma'^2}} T \\ &\quad + \frac{\eta\sigma' - (1 - \eta)\sqrt{p^2 + q^2}}{\sqrt{2(1 - \eta + \eta^2)} \sqrt{p^2 + q^2 + \sigma'^2}} N \\ &\quad + \frac{(\eta\sqrt{p^2 + q^2} + (1 - \eta)\sigma') \sin \sigma + \sqrt{p^2 + q^2 + \sigma'^2} \cos \sigma}{\sqrt{2(1 - \eta + \eta^2)} \sqrt{p^2 + q^2 + \sigma'^2}} B.\end{aligned}\quad (2.29)$$

Considering the equations (2.28) and (2.29) it is easily seen

$$\begin{aligned}\beta_4 \wedge T_{\beta_4} &= \frac{((2 - \eta)\sqrt{p^2 + q^2} - (1 + \eta)\sigma') \cos \sigma + (2\eta)\sqrt{p^2 + q^2 + \sigma'^2} \sin \sigma}{\sqrt{6 - 6\eta + 6\eta^2} \sqrt{p^2 + q^2 + \sigma'^2}} T \\ &\quad + \frac{(2 - \eta)\sigma' + (1 + \eta)\sqrt{p^2 + q^2}}{\sqrt{6 - 6\eta + 6\eta^2} \sqrt{p^2 + q^2 + \sigma'^2}} N \\ &\quad + \frac{((\eta - 2)\sqrt{p^2 + q^2} - (1 + \eta)\sigma') \sin \sigma + (2\eta - 1)\sqrt{p^2 + q^2 + \sigma'^2} \cos \sigma}{\sqrt{6 - 6\eta + 6\eta^2} \sqrt{p^2 + q^2 + \sigma'^2}} B.\end{aligned}\quad (2.30)$$

Differentiating (2.29) where

$$\begin{cases} \bar{\rho}_1 = -2 + 4\frac{1}{\eta} - 4\frac{1}{\eta^2} + 2\frac{1}{\eta^3} + 2\frac{1}{\eta'}(2\frac{1}{\eta} - 1) \\ \bar{\rho}_2 = -2 + 2\frac{1}{\eta} - 4\frac{1}{\eta^2} + 2\frac{1}{\eta^3} - 2\frac{1}{\eta^4} - \frac{1}{\eta}(1 + \frac{1}{\eta}) \\ \bar{\rho}_3 = 2\frac{1}{\eta} - 4\frac{1}{\eta^2} + 4\frac{1}{\eta^3} - 2\frac{1}{\eta^4} + \frac{1}{\eta'}(2 - \frac{1}{\eta}) \end{cases}\quad (2.31)$$

including we can write

$$\begin{aligned}T'_{\beta_4} &= \frac{(\bar{\rho}_1\sqrt{p^2 + q^2} + \bar{\rho}_2\sigma')\eta^4\sqrt{3}\cos \sigma + \bar{\rho}_3\eta^4\sqrt{3p^2 + q^2 + 3\sigma'^2} \sin \sigma}{4(1 - \eta + \eta^2)^2 \sqrt{\sigma'^2 + p^2 + q^2}} T \\ &\quad + \frac{(\bar{\rho}_1\sigma' - \bar{\rho}_2\sqrt{p^2 + q^2})\eta^4\sqrt{3}}{4(1 - \eta + \eta^2)^2 \sqrt{\sigma'^2 + p^2 + q^2}} N \\ &\quad + \frac{\bar{\rho}_3\eta^4\sqrt{3p^2 + q^2 + 3\sigma'^2} \cos \sigma - (\bar{\rho}_1\sqrt{p^2 + q^2} + \bar{\rho}_2\sigma')\eta^4\sqrt{3}\sin \sigma}{4(1 - \eta + \eta^2)^2 \sqrt{\sigma'^2 + p^2 + q^2}} B,\end{aligned}\quad (2.32)$$

where,  $\kappa_g^{\beta_4}$  geodesic curvature for Mannheim curve  $\beta_4(s_{\beta_4})$  is

$$\kappa_g^{\beta_4} = \frac{(2\eta^4 - \eta^5)\bar{\rho}_1 - (\eta^5 + \eta^4)\bar{\rho}_2 + (2\eta^5 - \eta^4)\bar{\rho}_3}{4\sqrt{2}(1 - \eta + \eta^2)^{\frac{5}{2}}}. \quad (2.33)$$

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## F-Root Square Mean Labeling of Graphs Obtained From Paths

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**Abstract:** A function  $f$  is called a  $F$ -root square mean labeling of a graph  $G(V, E)$  with  $p$  vertices and  $q$  edges if  $f : V(G) \rightarrow \{1, 2, 3, \dots, q+1\}$  is injective and the induced function  $f^*$  defined as

$$f^*(uv) = \left\lfloor \sqrt{\frac{f(u)^2 + f(v)^2}{2}} \right\rfloor$$

for all  $uv \in E(G)$ , is bijective. A graph that admits a  $F$ -root square mean labeling is called a  $F$ -root square mean graph. In this paper, we study the  $F$ -root square meanness of the path  $P_n$ , the graph  $P_n \circ S_m$ , the graph  $P_n \circ K_2$ , the graph  $TW(P_n)$ , the graph  $[P_n; S_m]$ , the graph  $S(P_n \circ K_1)$ , the graph  $M(P_n)$ , the graph  $T(P_n)$ , the graph  $P_n^2$ , the ladder graph  $L_n$  and the slanting ladder graph  $SL_n$ .

**Key Words:** Labeling, Smarandache power root mean labeling,  $F$ -root square mean labeling,  $F$ -root square mean graph.

**AMS(2010):** 05C12.

### §1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let  $G(V, E)$  be a graph with  $p$  vertices and  $q$  edges. For notations and terminology, we follow [3]. For a detailed survey on graph labeling, we refer [2].

Path on  $n$  vertices is denoted by  $P_n$ . A star graph  $S_n$  is the complete bipartite graph  $K_{1,n}$ . The graph  $G \circ S_m$  is obtained from  $G$  by attaching  $m$  pendant vertices to each vertex of  $G$ . A Twig  $TW(P_n)$ ,  $n \geq 4$  is a graph obtained from a path by attaching exactly two pendant vertices to each internal vertices of the path  $P_n$ . If  $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \dots, v_{m+1}^{(i)}$  and  $u_1, u_2, u_3, \dots, u_n$  be the vertices of  $i^{th}$  copy of the star graph  $S_m$  and the path  $P_n$  respectively, then the graph  $[P_n; S_m]$  is obtained from  $n$  copies of  $S_m$  and the path  $P_n$  by joining  $u_i$  with the central vertex  $v_1^{(i)}$  of the  $i^{th}$  copy of  $S_m$  by means of an edge, for  $1 \leq i \leq n$ .

A subdivision of a graph  $G$ , denoted by  $S(G)$ , is a graph obtained by subdividing edge of

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<sup>1</sup>Received May 10, 2016, Accepted May 25, 2017.

$G$  by a vertex.

The middle graph  $M(G)$  of a graph  $G$  is the graph whose vertex set is  $\{v : v \in V(G)\} \cup \{e : e \in E(G)\}$  and the edge set is  $\{e_1e_2 : e_1, e_2 \in E(G)$  and  $e_1$  and  $e_2$  are adjacent edges of  $G\} \cup \{ve : v \in V(G), e \in E(G)$  and  $e$  is incident with  $v\}$ . The total graph  $T(G)$  of a graph  $G$  is the graph whose vertex set is  $V(G) \cup E(G)$  and two vertices are adjacent if and only if either they are adjacent vertices of  $G$  or adjacent edges of  $G$  or one is a vertex of  $G$  and the other one is an edge incident on it. Square of a graph  $G$ , denoted by  $G^2$ , has the vertex set as in  $G$  and two vertices are adjacent in  $G^2$  if they are at a distance either 1 or 2 apart in  $G$ . Let  $G_1$  and  $G_2$  be any two graphs with  $p_1$  and  $p_2$  vertices respectively. Then the *Cartesian product*  $G_1 \times G_2$  has  $p_1p_2$  vertices which are  $\{(u, v) : u \in G_1, v \in G_2\}$  and the edges are obtained as follows:  $(u_1, v_1)$  and  $(u_2, v_2)$  are adjacent in  $G_1 \times G_2$  if either  $u_1 = u_2$  and  $v_1$  and  $v_2$  are adjacent in  $G_2$  or  $u_1$  and  $u_2$  are adjacent in  $G_1$  and  $v_1 = v_2$ . A ladder graph  $L_n$  is the graph  $P_2 \times P_n$ .

The slanting ladder  $SL_n$  is a graph obtained from two paths  $u_1, u_2, \dots, u_n$  and  $v_1, v_2, \dots, v_n$  by joining each  $v_i$ , with  $u_{i+1}, 1 \leq i \leq n-1$ .

The concept of root square mean labeling was introduced and studied by S.S. Sandhya et al. in [4,5]. Motivated by the works of so many authors in the area of graph labeling, we introduce a new type of labeling called *F-root square mean labeling*.

A labeling  $f$  on a graph  $G(V, E)$  with  $p$  vertices and  $q$  edges is called a *Smarandache power root mean labeling* for an integer  $m \geq 1$  if  $f : V(G) \rightarrow \{1, 2, 3, \dots, q+1\}$  is injective and the induced function  $f^*$  defined by

$$f^*(uv) = \left\lfloor \sqrt[m]{\frac{f(u)^m + f(v)^m}{2}} \right\rfloor$$

is bijective for all  $uv \in E(G)$ . Particularly, if  $m = 1$ , such a Smarandache power root mean labeling is nothing else but the mean labeling and  $m = 2$  is called the *F-root square mean labeling* on graph  $G(V, E)$  with an injective  $f : V(G) \rightarrow \{1, 2, 3, \dots, q+1\}$  and an induced bijective

$$f^*(uv) = \left\lfloor \sqrt{\frac{f(u)^2 + f(v)^2}{2}} \right\rfloor$$

for all  $uv \in E(G)$ . A graph that admits a *F-root square mean labeling* is called a *F-root square mean graph*.

In [4], S.S. Sandhya et al. defined the root square mean labeling as follows:

A graph  $G(V, E)$  with  $p$  vertices and  $q$  edges is said to be Root Square Mean graph if it is possible to label the vertices  $x \in V$  with distinct labels  $f(x)$  from  $1, 2, 3, \dots, q+1$  in such a way that when each edge  $e = uv$  is labeled with

$$\left\lfloor \sqrt{\frac{f(u)^2 + f(v)^2}{2}} \right\rfloor \text{ or } \left\lceil \sqrt{\frac{f(u)^2 + f(v)^2}{2}} \right\rceil,$$

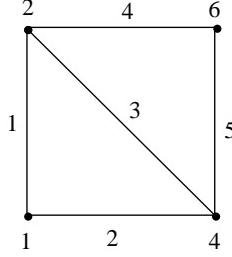
then the edge labels are distinct. In this case  $f$  is called a root square mean labeling of  $G$ .

In the above definition, the readers will get some confusion in finding the edge labels which

edge is assigned by flooring function and which edge is assigned by ceiling function. To avoid the confusion of assigning the edge labels in their definition, we just consider the flooring function

$$\left\lfloor \sqrt{\frac{f(u)^2 + f(v)^2}{2}} \right\rfloor$$

for our discussion. Based on our definition, a  $F$ -root square mean labeling of the graph  $K_5 - e$  is given in Figure 1.



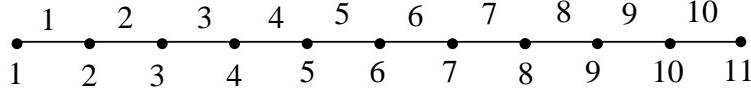
**Figure 1.** A  $F$ -root square mean labeling of  $K_4 - e$ .

In this paper, we study the  $F$ -root square meanness of the path  $P_n$ , the graph  $P_n \circ S_m$ , the graph  $P_n \circ K_2$ , the twig graph  $TW(P_n)$ , the graph  $[P_n; S_m]$ , the graph  $S(P_n \circ K_1)$ , the middle graph  $M(P_n)$ , the total graph  $T(P_n)$ , the square graph  $P_n^2$ , the ladder graph  $L_n$  and the slanting ladder graph  $SL_n$ .

## §2. Main Results

**Theorem 2.1** *Every path  $P_n$  is a  $F$ -root square mean graph.*

*Proof* Let  $v_1, v_2, v_3, \dots, v_n$  be the vertices of the path  $P_n$ . Define  $f : V(G) \rightarrow \{1, 2, 3, \dots, n\}$  as  $f(v_i) = i$ , for  $1 \leq i \leq n$ . Then the induced edge labeling is  $f^*(v_i v_{i+1}) = i$ , for  $1 \leq i \leq n-1$ . Hence  $f$  is a  $F$ -Root Square Mean labeling of path  $P_n$ . Thus the path  $P_n$  is a  $F$ -root square mean graph.  $\square$



**Figure 2.** A  $F$ -root square mean labeling of  $P_{11}$ .

**Theorem 2.2** *The graph  $P_n \circ S_m$  is a  $F$ -root square mean graph for  $n \geq 1$  and  $m \leq 4$ .*

*Proof* Let  $v_1, v_2, v_3, \dots, v_n$  be the vertices of the path  $P_n$  and  $u_1^{(i)}, u_2^{(i)}, u_3^{(i)}, \dots, u_m^{(i)}$  be the pendant vertices at each  $v_i$ , for  $1 \leq i \leq n$ .

**Case 1.**  $m = 1$ .

Define  $f : V(P_n \circ S_1) \rightarrow \{1, 2, 3, \dots, 2n\}$  as follows

$$\begin{aligned} f(v_i) &= 2i - 1, \text{ for } 1 \leq i \leq n, \\ f(u_1^{(i)}) &= 2i, \text{ for } 1 \leq i \leq n. \end{aligned}$$

Then the induced edge labeling is obtained as follows

$$\begin{aligned} f^*(v_i v_{i+1}) &= 2i, \text{ for } 1 \leq i \leq n-1, \\ f^*(v_i u_1^{(i)}) &= 2i-1, \text{ for } 1 \leq i \leq n. \end{aligned}$$

**Case 2.**  $m = 2$ .

Define  $f : V(P_n \circ S_2) \rightarrow \{1, 2, 3, \dots, 3n\}$  as follows

$$\begin{aligned} f(v_i) &= 3i - 1, \text{ for } 1 \leq i \leq n, \\ f(u_1^{(i)}) &= 3i - 2, \text{ for } 1 \leq i \leq n, \\ f(u_2^{(i)}) &= 3i, \text{ for } 1 \leq i \leq n. \end{aligned}$$

Then the induced edge labeling is obtained as follows

$$\begin{aligned} f^*(v_i v_{i+1}) &= 3i, \text{ for } 1 \leq i \leq n-1, \\ f^*(v_i u_1^{(i)}) &= 3i-2, \text{ for } 1 \leq i \leq n, \\ f^*(v_i u_2^{(i)}) &= 3i-1, \text{ for } 1 \leq i \leq n. \end{aligned}$$

**Case 3.**  $m = 3$ .

Define  $f : V(P_n \circ S_3) \rightarrow \{1, 2, 3, \dots, 4n\}$  as follows

$$\begin{aligned} f(v_i) &= 4i - 2, \text{ for } 1 \leq i \leq n, \\ f(u_1^{(i)}) &= 4i - 3, \text{ for } 1 \leq i \leq n, \\ f(u_2^{(i)}) &= 4i - 1, \text{ for } 1 \leq i \leq n, \\ f(u_3^{(i)}) &= 4i, \text{ for } 1 \leq i \leq n. \end{aligned}$$

Then the induced edge labeling is obtained as follows

$$\begin{aligned} f^*(v_i v_{i+1}) &= 4i, \text{ for } 1 \leq i \leq n-1, \\ f^*(v_i u_1^{(i)}) &= 4i-3, \text{ for } 1 \leq i \leq n, \\ f^*(v_i u_2^{(i)}) &= 4i-2, \text{ for } 1 \leq i \leq n, \\ f^*(v_i u_3^{(i)}) &= 4i-1, \text{ for } 1 \leq i \leq n. \end{aligned}$$

**Case 4.**  $m = 4$ .

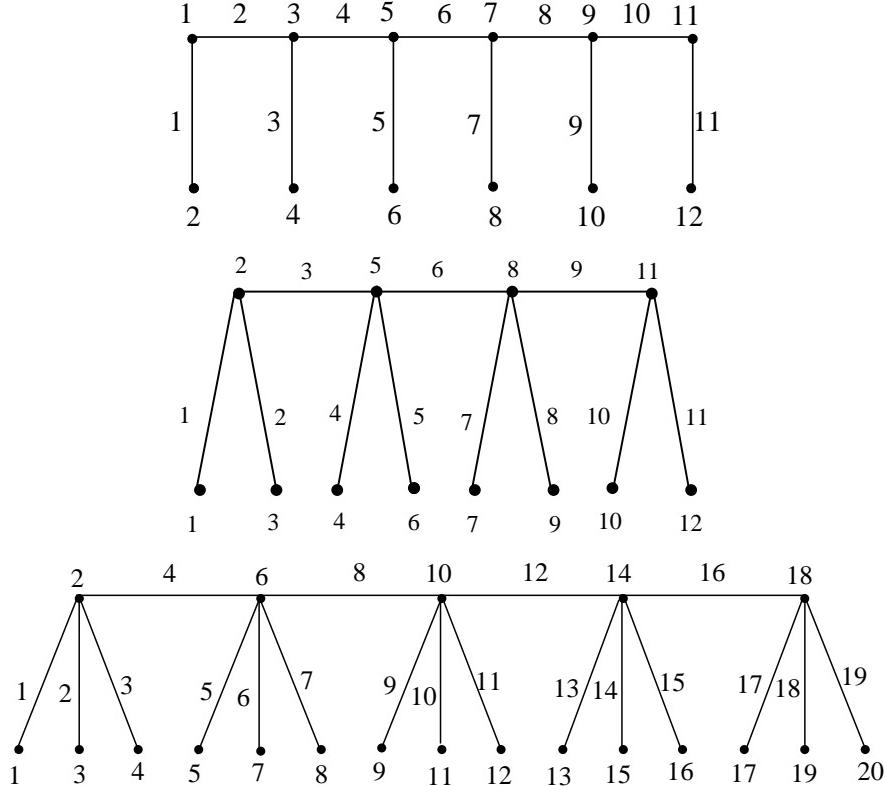
Define  $f : V(P_n \circ S_4) \rightarrow \{1, 2, 3, \dots, 5n\}$  as follows

$$\begin{aligned} f(v_1) &= 2, \quad f(v_i) = 5i - 2, \text{ for } 1 \leq i \leq n, \quad f(u_1^{(1)}) = 1, \\ f(u_1^{(i)}) &= 5i - 5, \text{ for } 2 \leq i \leq n, \quad f(u_2^{(1)}) = 3, \\ f(u_2^{(i)}) &= 5i - 3, \text{ for } 2 \leq i \leq n, \\ f(u_3^{(i)}) &= 5i - 1, \text{ for } 1 \leq i \leq n, \\ f(u_4^{(i)}) &= 5i + 1, \text{ for } 1 \leq i \leq n-1, \quad f(u_4^{(n)}) = 5n. \end{aligned}$$

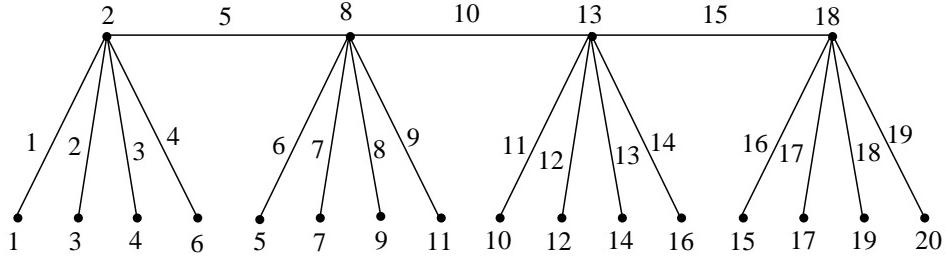
Then the induced edge labeling is obtained as follows

$$\begin{aligned} f^*(v_i v_{i+1}) &= 5i, \text{ for } 1 \leq i \leq n-1, \quad f^*(v_i u_1^{(i)}) = 5i - 4, \text{ for } 1 \leq i \leq n, \\ f^*(v_i u_2^{(i)}) &= 5i - 3, \text{ for } 1 \leq i \leq n, \quad f^*(v_i u_3^{(i)}) = 5i - 2, \text{ for } 1 \leq i \leq n, \\ f^*(v_i u_4^{(i)}) &= 5i - 1, \text{ for } 1 \leq i \leq n. \end{aligned}$$

Hence,  $f$  is a  $F$ -root square mean labeling of the graph  $P_n \circ S_m$ . Thus the graph  $P_n \circ S_m$  is a  $F$ -root square mean graph for  $n \geq 1$  and  $m \leq 4$ .  $\square$



**Figure 3.** A  $F$ -root square mean labeling of  $P_6 \circ S_1$ ,  $P_4 \circ S_2$  and  $P_5 \circ S_3$ .



**Figure 4.** A  $F$ -root square mean labeling of  $P_4 \circ S_4$ .

**Theorem 2.3** *The graph  $P_n \circ K_2$  is a  $F$ -root square mean graph for  $n \geq 1$ .*

*Proof* Let  $v_1, v_2, v_3, \dots, v_n$  be the vertices of the path  $P_n$  and  $u_i^{(1)}, u_i^{(2)}$  be the vertices of  $i^{th}$  copy of  $K_2$  attached with  $v_i$ , for  $1 \leq i \leq n$ .

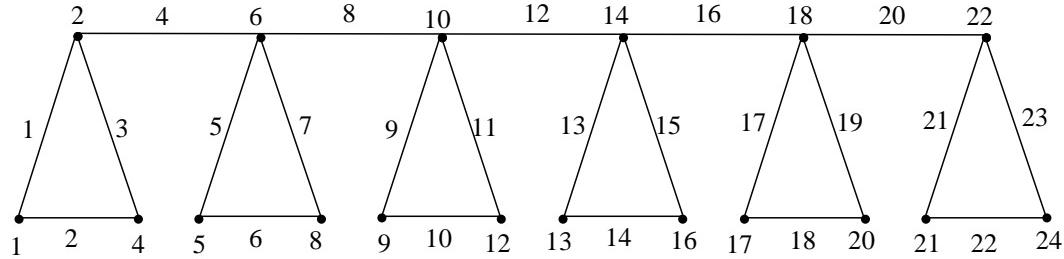
Define  $f : V(P_n \circ K_2) \rightarrow \{1, 2, 3, \dots, 4n\}$  as follows

$$\begin{aligned} f(v_i) &= 4i - 2, \text{ for } 1 \leq i \leq n, \\ f(u_i^{(1)}) &= 4i - 3, \text{ for } 1 \leq i \leq n, \\ f(u_i^{(2)}) &= 4i, \text{ for } 1 \leq i \leq n. \end{aligned}$$

Then the induced edge labeling is obtained as follows

$$\begin{aligned} f^*(v_i v_{i+1}) &= 4i, \text{ for } 1 \leq i \leq n-1, \\ f^*(u_i^{(1)} u_i^{(2)}) &= 4i-2, \text{ for } 1 \leq i \leq n, \\ f^*(u_i^{(1)} v_1) &= 4i-3, \text{ for } 1 \leq i \leq n, \\ f^*(u_i^{(2)} v_i) &= 4i-1, \text{ for } 1 \leq i \leq n. \end{aligned}$$

Hence  $f$  is a  $F$ -root square mean labeling of the graph  $P_n \circ K_2$ . Thus the graph  $P_n \circ K_2$  is a  $F$ -root square mean graph for  $n \geq 1$ .  $\square$



**Figure 5.** A  $F$ -root square mean labeling of  $P_6 \circ K_2$ .

**Theorem 2.4** *The twig graph  $TW(P_n)$  of the path  $P_n$  is a  $F$ -root square mean graph for  $n \geq 4$ .*

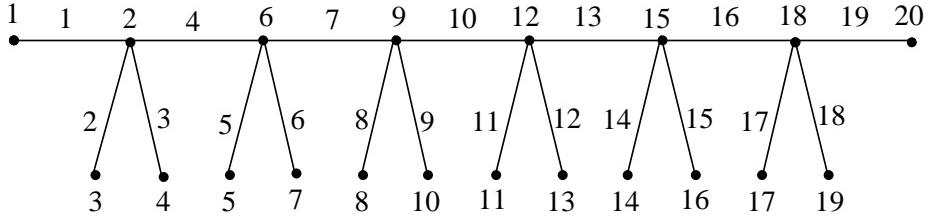
*Proof* Let  $v_1, v_2, v_3, \dots, v_n$  be the vertices of the path  $P_n$  and  $u_1^{(i)}, u_2^{(i)}$  be the pendant vertices at each vertex  $v_i$ , for  $2 \leq i \leq n - 1$ . Define  $f : V(TW(P_n)) \rightarrow \{1, 2, 3, \dots, 3n - 4\}$  as follows

$$\begin{aligned} f(v_1) &= 1, \quad f(v_2) = 2, \quad f(v_i) = 3i - 3, \text{ for } 3 \leq i \leq n - 1, \\ f(v_n) &= 3n - 4, \quad f(u_1^{(2)}) = 3, \quad f(u_1^{(i)}) = 3i - 4, \text{ for } 3 \leq i \leq n - 1, \\ f(u_2^{(i)}) &= 3i - 2, \text{ for } 2 \leq i \leq n - 1. \end{aligned}$$

Then the induced edge labeling is obtained as follows

$$\begin{aligned} f^*(v_i v_{i+1}) &= 3i - 2, \text{ for } 1 \leq i \leq n - 1, \\ f^*(v_i u_1^{(i)}) &= 3i - 4, \text{ for } 2 \leq i \leq n - 1 \\ f^*(v_i u_2^{(i)}) &= 3i - 3, \text{ for } 2 \leq i \leq n - 1. \end{aligned}$$

Hence  $f$  is a  $F$ -root square mean labeling of the graph  $TW(P_n)$ . Thus the graph  $TW(P_n)$  is a  $F$ -root square mean graph for  $n \geq 1$ .  $\square$



**Figure 6.** A  $F$ -root square mean labeling of  $TW(P_8)$ .

**Theorem 2.5** The graph  $[P_n; S_m]$  is a  $F$ -root square mean graph for  $m \leq 2$  and  $n \geq 1$ .

*Proof* Let  $u_1, u_2, u_3, \dots, u_n$  be the vertices of the path  $P_n$  and  $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \dots, v_{m+1}^{(i)}$  be the vertices of the star graph  $S_m$  such that  $v_1^{(i)}$  is the central vertex of the star graph  $S_m$ ,  $1 \leq i \leq n$ .

**Case 1.**  $m = 1$

Define  $f : V([P_n; S_1]) \rightarrow \{1, 2, 3, \dots, 3n\}$  as follows

$$\begin{aligned} f(u_i) &= \begin{cases} 3i, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 3i - 2, & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases} \\ f(v_1^{(i)}) &= 3i - 1, \text{ for } 1 \leq i \leq n, \\ f(v_2^{(i)}) &= \begin{cases} 3i - 2, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 3i, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases} \end{aligned}$$

Then the induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = 3i, \text{ for } 1 \leq i \leq n-1,$$

$$f^*(u_i v_1^{(i)}) = \begin{cases} 3i-1, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 3i-2, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_1^{(i)} v_2^{(i)}) = \begin{cases} 3i-2, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 3i-1, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

**Case 2.**  $m = 2$

Define  $f : V([P_n; S_2]) \rightarrow \{1, 2, 3, \dots, 4n\}$  as follows

$$f(u_i) = \begin{cases} 4i-1, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4i-3, & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_1^{(i)}) = 4i-2, \text{ for } 1 \leq i \leq n,$$

$$f(v_2^{(i)}) = \begin{cases} 4i-3, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4i-1, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f(v_3^{(i)}) = 4i, \text{ for } 1 \leq i \leq n.$$

Then the induced edge labeling is obtained as follows

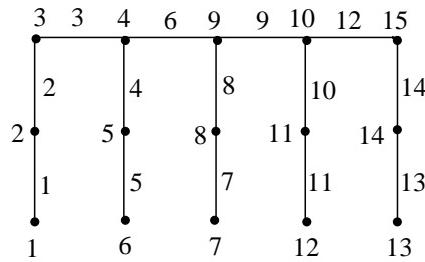
$$f^*(u_i u_{i+1}) = 4i, \text{ for } 1 \leq i \leq n-1,$$

$$f^*(u_i v_1^{(i)}) = \begin{cases} 4i-2, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4i-3, & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

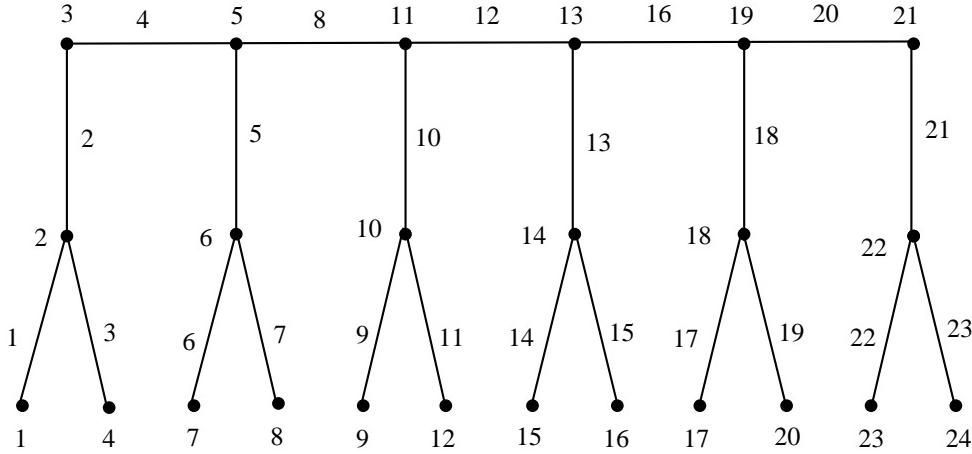
$$f^*(v_1^{(i)} v_2^{(i)}) = \begin{cases} 4i-3, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 4i-2, & 1 \leq i \leq n \text{ and } i \text{ is even} \end{cases}$$

$$f^*(v_1^{(i)} v_3^{(i)}) = 4i-1, \text{ for } 1 \leq i \leq n.$$

Hence  $f$  is a  $F$ -root square mean labeling of the graph  $[P_n; S_m]$ . Thus the graph  $[P_n; S_m]$  is a  $F$ -root square mean graph for  $n \geq 1$ .  $\square$



**Figure 7.** A  $F$ -root square mean labeling of  $[P_5; S_1]$ .



**Figure 8.** A  $F$ -root square mean labeling of  $[P_6; S_2]$ .

**Theorem 2.6** *The graph  $S(P_n \circ K_1)$  is a  $F$ -root square mean graph for  $n \geq 1$ .*

*Proof* In  $P_n \circ K_1$ , let  $u_i, 1 \leq i \leq n$ , be the vertices on the path  $P_n$  and  $v_i$  be the vertex attached at each vertex  $u_i, 1 \leq i \leq n$ .

Let  $x_i$  be the vertex which divides the edge  $u_i v_i$ , for  $1 \leq i \leq n$  and  $y_i$  be the vertex which divides the edge  $u_i u_{i+1}$ , for  $1 \leq i \leq n-1$ . Then,

$$\begin{aligned} V(S(P_n \circ K_1)) &= \{u_i, v_i, x_i, y_j; 1 \leq i \leq n \text{ and } 1 \leq j \leq n-1\}, \\ E(S(P_n \circ K_1)) &= \{u_i x_i, v_i y_i; 1 \leq i \leq n\} \bigcup \{u_i y_i, y_i u_{i+1}; 1 \leq i \leq n-1\}. \end{aligned}$$

Define  $f : V(S(P_n \circ K_1)) \rightarrow \{1, 2, 3, \dots, 4n-1\}$  as follows

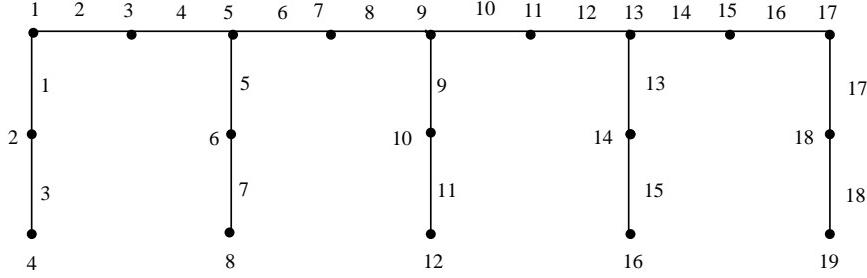
$$\begin{aligned} f(u_i) &= 4i-3, \text{ for } 1 \leq i \leq n, \\ f(y_i) &= 4i-1, \text{ for } 1 \leq i \leq n-1, \\ f(x_i) &= 4i-2, \text{ for } 1 \leq i \leq n, \\ f(v_i) &= 4i, \text{ for } 1 \leq i \leq n-1, \\ f(v_n) &= 4n-1. \end{aligned}$$

Then the induced edge labeling is obtained as follows

$$\begin{aligned} f^*(u_i y_i) &= 4i-2, \text{ for } 1 \leq i \leq n-1, \\ f^*(y_i u_{i+1}) &= 4i, \text{ for } 1 \leq i \leq n-1, \\ f^*(u_i x_i) &= 4i-3, \text{ for } 1 \leq i \leq n, \\ f^*(x_i v_i) &= 4i-1, \text{ for } 1 \leq i \leq n-1, \\ f^*(v_n v_n) &= 4n-2. \end{aligned}$$

Hence  $f$  is a  $F$ -root square mean labeling of  $S(P_n \circ K_1)$ . Thus the graph  $S(P_n \circ K_1)$  is a

*F*-root square mean graph for  $n \geq 1$ .  $\square$



**Figure 9.** A *F*-root square mean labeling of  $S(P_5 \circ K_1)$ .

**Theorem 2.7** *The middle graph  $M(P_n)$  of a path  $P_n$  is a *F*-root square mean graph.*

*Proof* Let  $V(P_n) = \{v_1, v_2, v_3, \dots, v_n\}$  and  $E(P_n) = \{e_i = v_i v_{i+1}; 1 \leq i \leq n-1\}$  be the vertex set and edge set of the path  $P_n$ . Then,

$$\begin{aligned} V(M(P_n)) &= \{v_1, v_2, v_3, \dots, v_n, e_1, e_2, e_3, \dots, e_{n-1}\}, \\ E(M(P_n)) &= \{v_i e_i, e_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{e_i e_{i+1}; 1 \leq i \leq n-2\}. \end{aligned}$$

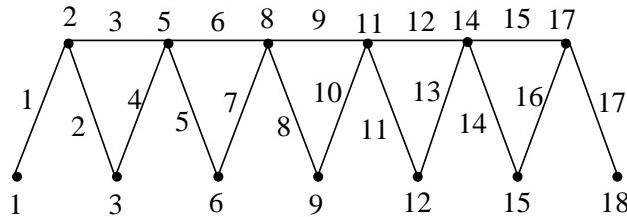
Define  $f : V(M(P_n)) \rightarrow \{1, 2, 3, \dots, 3n-3\}$  as follows

$$\begin{aligned} f(v_i) &= \begin{cases} 1, & i = 1 \\ 3i-3, & 2 \leq i \leq n, \end{cases} \\ f(e_i) &= 3i-1, \text{ for } 1 \leq i \leq n-1. \end{aligned}$$

Then the induced edge labeling is obtained as follows

$$\begin{aligned} f^*(v_i e_i) &= 3i-2, \text{ for } 1 \leq i \leq n-1, \\ f^*(e_i v_{i+1}) &= 3i-1, \text{ for } 1 \leq i \leq n-1, \\ f^*(e_i e_{i+1}) &= 3i, \text{ for } 1 \leq i \leq n-2. \end{aligned}$$

Hence  $f$  is a *F*-root square mean labeling of  $M(P_n)$ . Thus the graph  $M(P_n)$  is a *F*-root square mean graph.  $\square$



**Figure 10.** A *F*-root square mean labeling of  $M(P_7)$ .

**Theorem 2.8** *The total graph  $T(P_n)$  of a path  $P_n$  is a  $F$ -root square mean graph for  $n \geq 1$ .*

*Proof* Let  $V(P_n) = \{v_1, v_2, v_3, \dots, v_n\}$  and  $E(P_n) = \{e_i = v_i v_{i+1}; 1 \leq i \leq n-1\}$  be the vertex set and edge set of the path  $P_n$ . Then,

$$\begin{aligned} V(T(P_n)) &= \{v_1, v_2, v_3, \dots, v_n, e_1, e_2, e_3, \dots, e_{n-1}\} \\ E(T(P_n)) &= \{v_i v_{i+1}, e_i v_i, e_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{e_i e_{i+1}; 1 \leq i \leq n-2\}. \end{aligned}$$

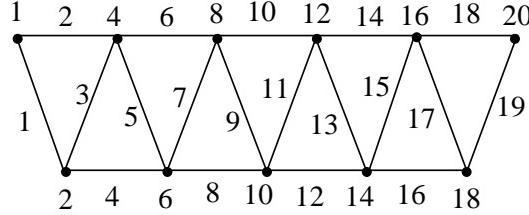
Define  $f : V(T(P_n)) \rightarrow \{1, 2, 3, \dots, 4(n-1)\}$  as follows

$$\begin{aligned} f(v_1) &= 1, & f(v_i) &= 4i - 4, \text{ for } 2 \leq i \leq n \\ f(e_i) &= 4i - 2, \text{ for } 1 \leq i \leq n-1. \end{aligned}$$

Then the induced edge labeling is obtained as follows

$$\begin{aligned} f^*(v_i v_{i+1}) &= 4i - 2, \text{ for } 1 \leq i \leq n-1, & f^*(e_i e_{i+1}) &= 4i, \text{ for } 1 \leq i \leq n-2, \\ f^*(v_i e_i) &= 4i - 3, \text{ for } 1 \leq i \leq n-1 \\ f^*(e_i v_{i+1}) &= 4i - 1, \text{ for } 1 \leq i \leq n-1. \end{aligned}$$

Hence  $f$  is a  $F$ -root square mean labeling of  $T(P_n)$ . Thus the graph  $T(P_n)$  is a  $F$ -root square mean graph.  $\square$



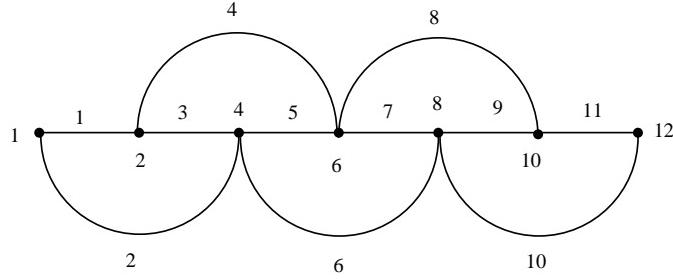
**Figure 11.** A  $F$ -root square mean labeling of  $T(P_6)$ .

**Theorem 2.9** *The square graph  $P_n^2$  of the path  $P_n$  is a  $F$ -root square mean graph for  $n \geq 1$ .*

*Proof* Let  $v_1, v_2, v_3, \dots, v_n$  be the vertices of the path  $P_n$ . Define  $f : V(P_n^2) \rightarrow \{1, 2, 3, \dots, 2(n-1)\}$  as  $f(v_1) = 1$  and  $f(v_i) = 2i - 2$  for  $2 \leq i \leq n$ . Then the induced edge labeling is obtained as follows

$$\begin{aligned} f^*(v_i v_{i+1}) &= 2i - 1, \text{ for } 1 \leq i \leq n-1, \\ f^*(v_i v_{i+2}) &= 2i, \text{ for } 1 \leq i \leq n-2. \end{aligned}$$

Hence  $f$  is a  $F$ -root square mean labeling of  $P_n^2$ . Thus the graph  $P_n^2$  is a  $F$ -root square mean graph.  $\square$



**Figure 12.** A  $F$ -root square mean labeling of  $P_7^2$ .

**Theorem 2.10** *The ladder graph  $L_n$  is a  $F$ -root square mean graph for  $n \geq 1$ .*

*Proof* Let  $G = P_2 \times P_n$  be the ladder graph for any positive integer  $n$ , having  $2n$  vertices and  $3n - 2$  edges. Let  $u_1, u_2, \dots, u_n$  and  $v_1, v_2, \dots, v_n$  be the vertices of  $G$ . Then the edge set of  $G$  is  $\{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_i v_i : 1 \leq i \leq n\}$ .

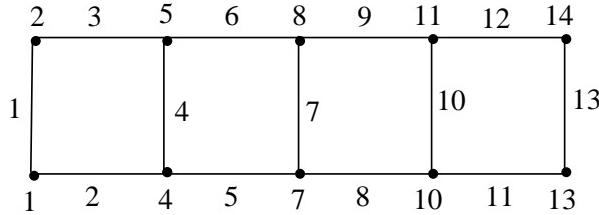
Define  $f : V(G) \rightarrow \{1, 2, 3, \dots, 3n - 1\}$  as follows

$$\begin{aligned} f(u_i) &= 3i - 1, \text{ for } 1 \leq i \leq n, \\ f(v_i) &= 3i - 2, \text{ for } 1 \leq i \leq n. \end{aligned}$$

Then the induced edge labeling is obtained as follows

$$\begin{aligned} f^*(u_i u_{i+1}) &= 3i, \text{ for } 1 \leq i \leq n - 1, \\ f^*(v_i v_{i+1}) &= 3i - 1, \text{ for } 1 \leq i \leq n - 1 \\ f^*(u_i v_i) &= 3i - 2, \text{ for } 1 \leq i \leq n. \end{aligned}$$

Hence  $f$  is a  $F$ -root square mean labeling of  $L_n$ . Thus the graph  $L_n$  is a  $F$ -root square mean graph.  $\square$



**Figure 13.** A  $F$ -root square mean labeling of  $L_5$ .

**Theorem 2.11** *The slanting ladder graph  $SL_n$  is a  $F$ -root square mean graph for  $n \geq 2$ .*

*Proof* Let the vertex set of  $SL_n$  be  $\{u_1, u_2, u_3, \dots, u_n, v_1, v_2, v_3, \dots, v_n\}$  and the edge set of  $SL_n$  be  $\{u_i u_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_i u_{i+1} : 1 \leq i \leq n - 1\}$ . Then  $SL_n$  has  $2n$  vertices and  $3n - 3$  edges.

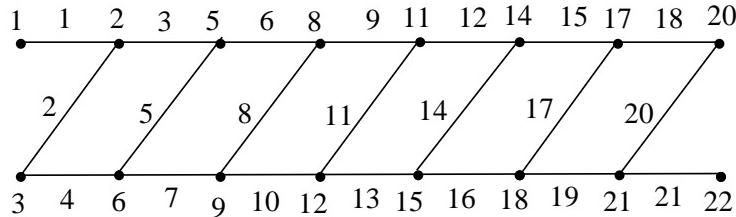
Define  $f : V(SL_n) \rightarrow \{1, 2, 3, \dots, 3n - 2\}$  as follows

$$\begin{aligned} f(u_1) &= 1, & f(u_i) &= 3i - 4, \text{ for } 2 \leq i \leq n, \\ f(v_i) &= 3i, \text{ for } 1 \leq i \leq n-1 & \text{and } f(v_n) &= 3n - 2. \end{aligned}$$

Then the induced edge labeling is obtained as follows

$$\begin{aligned} f^*(u_i u_{i+1}) &= \begin{cases} 1, & i = 1 \\ 3i - 3, & 2 \leq i \leq n-1, \end{cases} \\ f^*(v_i v_{i+1}) &= 3i + 1, \text{ for } 1 \leq i \leq n-2, \\ f^*(v_{n-1} v_n) &= 3n - 3 \\ f^*(v_i u_{i+1}) &= 3i - 1, \text{ for } 1 \leq i \leq n-1. \end{aligned}$$

Hence  $f$  is a  $F$ -root square mean labeling of  $SL_n$ . Thus the graph  $SL_n$  is a  $F$ -root square mean graph.  $\square$



**Figure 14.** A  $F$ -root square mean labeling of  $SL_8$ .

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## Some More 4-Prime Cordial Graphs

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**Abstract:** Let  $G$  be a  $(p, q)$  graph,  $H \prec G$  and  $f : V(G) \rightarrow \{1, 2, \dots, k\}$  be a map. For each edge  $uv$ , assign the label  $\gcd(f(u), f(v))$ . Then,  $f$  is called Smarandachely  $k$ -prime cordial labeling on  $G$  to  $H$  if  $|v_f^H(i) - v_f^H(j)| \leq 1$ ,  $i, j \in \{1, 2, \dots, k\}$  and  $|e_f^H(0) - e_f^H(1)| \leq 1$ , but there exist integers  $0 \leq i \neq j \leq k$  such that  $|v_f^{G \setminus H}(i) - v_f^{G \setminus H}(j)| \geq 2$ , or  $|e_f^{G \setminus H}(0) - e_f^{G \setminus H}(1)| \geq 2$ , where  $v_f^H(x)$ ,  $v_f^{G \setminus H}(x)$  respectively denotes the numbers of vertices of  $H$ ,  $G \setminus H$  labeled with  $x$ ,  $e_f^H(1)$ ,  $e_f^H(0)$  and  $e_f^{G \setminus H}(1)$ ,  $e_f^{G \setminus H}(0)$  respectively denote the number of edges labeled with 1 and not labeled with 1 in  $H$ ,  $G \setminus H$ . Particularly, a Smarandachely  $k$ -prime cordial labeling on  $G$  to  $G$  is called  $k$ -prime cordial labeling with  $v_f(x)$ ,  $e_f(1)$  and  $e_f(0)$  replacing notations  $v_f^H(x)$ ,  $e_f^H(1)$  and  $e_f^H(0)$  for abbreviation. A graph with a  $k$ -prime cordial labeling is called a  $k$ -prime cordial graph. In this paper we investigate 4-prime cordial labeling behavior of lotus inside a circle, sunflower graph,  $S(K_2 + mK_1)$ ,  $S(P_n \odot K_1)$ , dodecahedron, and some more graphs.

**Key Words:** Smarandachely  $k$ -prime labelling,  $k$ -prime labelling, cycle, wheel, join, sunflower graph, lotus inside a circle.

**AMS(2010):** 05C78.

### §1. Introduction

In this paper graphs are finite, simple and undirected. Let  $G$  be a  $(p, q)$  graph where  $p$  refers the number of vertices of  $G$  and  $q$  refers the number of edge of  $G$ . The number of vertices of a graph  $G$  is called order of  $G$ , and the number of edges is called size of  $G$ . In 1987, Cahit introduced the concept of cordial labeling of graphs [1]. Sundaram, Ponraj, Somasundaram [5] have introduced the notion of prime cordial labeling of graphs. Also they discussed the prime cordial labeling behavior of various graphs. Recently Ponraj et al. [7], introduced  $k$ -prime cordial labeling of graphs. They have studied 3-prime cordiality of several graphs in [7, 8]. In [9, 10] Ponraj et al. studied the 4-prime cordial labeling behavior of complete graph,

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<sup>1</sup>Received September 8, 2016, Accepted May 26, 2017.

book, flower,  $mC_n$ , wheel, gear, double cone, helm, closed helm, butterfly graph, and friendship graph and some more graphs. In this paper we have studied about the 4-prime cordiality of lotus inside a circle, sunflower graph and some more graphs. Let  $x$  be any real number. Then  $\lfloor x \rfloor$  stands for the largest integer less than or equal to  $x$  and  $\lceil x \rceil$  stands for smallest integer greater than or equal to  $x$ . Terms not defined here follow from Harary [3] and Gallian [2].

## §2. Preliminaries

**Remark 2.1([6])** A 2-prime cordial labeling is a product cordial labeling.

**Remark 2.2([5])** A  $p$ -prime cordial labeling is a prime cordial labeling.

**Definition 2.3** *The join of two graphs  $G_1 + G_2$  is obtained from  $G_1$  and  $G_2$  and whose vertex set is  $V(G_1 + G_2) = V(G_1) \cup V(G_2)$  and edge set  $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$ .*

**Definition 2.4** *The graph  $C_n + K_1$  is called a wheel. In a wheel, the vertex of degree  $n$  is called the central vertex and the vertices on the cycle  $C_n$  are called rim vertices.*

**Definition 2.5** *The sunflower graph  $SF_n$  is obtained by taking a wheel with central vertex  $u$  and the cycle  $C_n : u_1u_2 \cdots u_nu_1$  and new vertices  $v_1, v_2, \dots, v_n$  where  $v_i$  is joined by vertices  $u_i, u_{i+1} \pmod{n}$ .*

**Definition 2.6** *The lotus inside a circle  $LC_n$  is a graph obtained from the cycle  $C_n : v_1v_2 \cdots v_nv_1$  and a star  $K_{1,n}$  with central vertex  $u$  and the end vertices  $u_1, u_2, \dots, u_n$  by joining each  $u_i$  to  $v_i$  and  $v_{i+1} \pmod{n}$ .*

**Definition 2.7** *The subdivision graph  $S(G)$  of a graph  $G$  is obtained by replacing each edge  $uv$  by a path  $uvw$ .*

**Definition 2.8** *The graph  $P_n^2$  is obtained from the path  $P_n$  by adding edges that joins all vertices  $u$  and  $v$  with  $d(u, v) = 2$ .*

**Definition 2.9** *Let  $G_1, G_2$  respectively be  $(p_1, q_1), (p_2, q_2)$  graphs. The corona of  $G_1$  with  $G_2$ ,  $G_1 \odot G_2$  is the graph obtained by taking one copy of  $G_1$  and  $p_1$  copies of  $G_2$  and joining the  $i^{th}$  vertex of  $G_1$  with an edge to every vertex in the  $i^{th}$  copy of  $G_2$ .*

**Definition 2.10** *The one-point union of  $t$  copies of the cycle  $C_3$  is called a friendship graph  $C_3^{(t)}$ .*

**Definition 2.11** *The  $DH_n$  is a graph with vertex set  $V(DH_n) = \{u_i, v_i, x_i, y_i : 1 \leq i \leq n\}$  and the edge set  $E(DH_n) = \{uu_{i+1}, y_iv_{i+1}, x_ix_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i, v_iy_i, x_iy_i : 1 \leq i \leq n\} \cup \{u_1u_n, y_nv_1, x_1x_n\}$ .  $DH_5$  is called a Tetrahedron.*

**Definition 2.12** *The Cartesian product graph  $G_1 \square G_2$  is defined as follows: Consider any two points  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$  in  $V = V_1 \times V_2$ . Then  $u$  and  $v$  are adjacent in  $G_1 \square G_2$*

whenever  $[u_1 = v_1 \text{ and } u_2v_2 \in E(G_2)]$  or  $[u_2 = v_2 \text{ and } u_1v_1 \in E(G_1)]$ .

**Definition 2.13** A ladder  $L_n$  is the graph  $P_n \times P_2$ . Let  $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$  and  $E(L_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{u_1u_n, v_1v_n\}$ .

The graph  $GL_n$  is obtained from the ladder  $L_n$  with  $V(GL_n) = V(L_n)$  and  $E(GL_n) = E(L_n) \cup \{u_iv_{i+1} : 1 \leq i \leq n-1\}$ .

**Theorem 2.14([7])** The cycle  $C_n$ ,  $n \neq 3$  is  $k$ -prime cordial where  $k$  is even.

### §3. Main Results

#### 3.1 Cycle Related Graphs

**Theorem 3.1** The lotus inside a circle  $LC_n$  is 4-prime cordial if and only if  $n > 4$ .

*Proof* Note that the order and size of  $LC_n$  are  $2n+1$  and  $4n$  respectively. Suppose  $n = 3$  or 4 then one can easily check that there does not exist a 4-prime cordial labeling and so we assume  $n > 4$ . Here we divide the proof into four cases.

**Case 1.**  $n \equiv 0 \pmod{4}$ .

We construct a labeling  $f$  as follows: assign the label 4 to the vertices  $v_1, v_2, \dots, v_{\frac{n}{2}}$  then put the label 3 to the vertex  $v_{\frac{n}{2}+1}$ . The remaining vertices of the cycle, namely,  $v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}, \dots, v_n$  are labeled by 1. Now we consider the center of the star  $u$ . The vertex  $u$  is labeled by 2. For the vertices  $u_1, u_2, \dots, u_n$ , first we consider the vertices  $u_{n-1}$  and  $u_n$ . Assign the labels 1, 2 respectively to the vertices  $u_{n-1}$  and  $u_n$ . Consider the vertices  $u_1, u_2, \dots, u_{\frac{n}{2}-1}$ . Fix the label 2 to this vertex. Then the vertices  $u_{\frac{n}{2}}, u_{\frac{n}{2}+1}, \dots, u_{n-2}$  are labeled by 3.

**Case 2.**  $n \equiv 1 \pmod{4}$ .

Assign the labels to the vertices  $u, u_i, 1 \leq i \leq n-3, v_j, 1 \leq j \leq n-1$  as in case 1. Then assign the labels 3, 1, 2 to the vertices  $u_{n-2}, u_{n-1}, u_n$  respectively. Finally we assign the label 4 to the vertex  $v_n$ .

**Case 3.**  $n \equiv 2 \pmod{4}$ .

Assign the label 4 to the vertices  $v_1, v_2, \dots, v_{\frac{n}{2}}$ . For the vertex  $v_{\frac{n}{2}+1}$  we assign the number 3. The remaining vertices of the cycle from  $v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}, \dots, v_n$  receives the label 1. Put the label 2 to the vertex  $u$ . Now we consider the vertices  $u_i, 1 \leq i \leq n$ . Assign the label 2 to the vertices  $u_n, u_i$  where  $1 \leq i \leq \frac{n}{2}-1$ , then assign the integer 3 to the vertices  $u_{\frac{n}{2}}, u_{\frac{n}{2}+1}, \dots, u_{n-2}$ . Finally we assign the label 1 to the vertex  $u_{n-1}$ .

**Case 4.**  $n \equiv 3 \pmod{4}$ .

First we consider the vertices of the cycle. The label 4 is used to the vertices  $v_i, 1 \leq i \leq \frac{n+1}{2}$ . Put the label 3 to the vertex  $v_{\frac{n+3}{2}}$ . The unlabeled vertices of the cycle are now labeled by 1. Then put the number 2 to the vertex  $u$ . If we consider the vertices  $u_i, 1 \leq i \leq n$ , we assign the label 2 to the vertices  $u_j$  where  $1 \leq j \leq \frac{n-1}{2}$  then assign 3 to the vertices  $u_{\frac{n+1}{2}}, \dots, u_{n-1}$ .

Finally put the number 1 to the vertex  $u_n$ . □

The following Table 1 establish that the above mentioned labeling  $f$  is a 4-prime cordial labeling.

Values of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod{4}$	$\lfloor \frac{2n+1}{4} \rfloor$	$\lceil \frac{2n+1}{4} \rceil$	$\lfloor \frac{2n+1}{4} \rfloor$	$\lceil \frac{2n+1}{4} \rceil$	$2n$	$2n$
$n \equiv 1 \pmod{4}$	$\lfloor \frac{2n+1}{4} \rfloor$	$\lceil \frac{2n+1}{4} \rceil$	$\lceil \frac{2n+1}{4} \rceil$	$\lceil \frac{2n+1}{4} \rceil$	$2n$	$2n$
$n \equiv 2 \pmod{4}$	$\lfloor \frac{2n+1}{4} \rfloor$	$\lceil \frac{2n+1}{4} \rceil$	$\lfloor \frac{2n+1}{4} \rfloor$	$\lceil \frac{2n+1}{4} \rceil$	$2n$	$2n$
$n \equiv 3 \pmod{4}$	$\lfloor \frac{2n+1}{4} \rfloor$	$\lceil \frac{2n+1}{4} \rceil$	$\lceil \frac{2n+1}{4} \rceil$	$\lceil \frac{2n+1}{4} \rceil$	$2n$	$2n$

**Table 1**

**Theorem 3.2** *The sunflower graph  $SF_n$  is 4-prime cordial for all  $n$ .*

*Proof* First we observe that the order and size of  $SF_n$  are  $2n+1$  and  $4n$  respectively. We consider the following cases.

**Case 1.**  $n \equiv 0 \pmod{4}$ .

Assign the label 2 to the vertices  $u_1, u_2, \dots, u_{\frac{n}{4}}$ . Then put the number 4 to the next consecutive vertices  $u_{\frac{n}{4}+1}, \dots, u_{\frac{n}{2}+1}$ . The next vertex  $u_{\frac{n}{2}+2}$  is labeled by 3. Then the remaining vertices of the cycle, namely,  $u_{\frac{n}{2}+3}, \dots, u_n$  are labeled by 1. For the central vertex  $u$ , we use the label 2. We now move to the vertices  $v_i$ ,  $1 \leq i \leq n$ . Assign the label 2 to the vertices  $v_1, v_2, \dots, v_{\frac{n}{4}-1}$ . Then assign the label 4 to the vertices  $v_{\frac{n}{4}}, v_{\frac{n}{4}+1}, \dots, v_{\frac{n}{2}-1}$ . The next three vertices  $v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}$  are labeled by 1, 3, 1 respectively. Finally the remaining unlabeled vertices received the integer 3.

**Case 2.**  $n \equiv 1 \pmod{4}$ .

First we consider the vertices of the cycle  $C_n$ . For the vertices  $u_1, u_2, \dots, u_{\frac{n-5}{4}}$ , we assign the label 2. The successive vertices  $u_{\frac{n-1}{4}}, \dots, u_{\frac{n-1}{2}}$  are labeled by 4. Put the label 3 to the vertex  $u_{\frac{n+1}{2}}$ . The vertices  $u_{\frac{n+3}{2}}, \dots, u_{n-1}$  are labeled by 1. Put the label 2 to the vertex  $u_n$ . Then assign the label 2 to the central vertex  $u$ . We now move to the vertices  $v_i$ ,  $1 \leq i \leq n$ . Assign the label 2 to the vertices  $v_1, v_2, \dots, v_{\frac{n-5}{4}}$ . Then put the number 4 to the vertices  $v_{\frac{n-1}{4}}, \dots, v_{\frac{n-3}{2}}$ . Put the labels 3, 1 respectively to the vertices  $v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}$ . Assign the label 3 to the vertices  $v_{\frac{n+3}{2}}, \dots, v_{n-1}$ . Finally we assign the number 2 to  $v_n$ .

**Case 3.**  $n \equiv 2 \pmod{4}$ .

We first consider the vertex  $u$ . Label it by 2. Then we consider the vertices of the cycle  $C_n$ . Assign the label 2 to the vertices  $u_1, u_2, \dots, u_{\frac{n+2}{4}}$ . Then put the integer 4 to the vertices  $u_{\frac{n+6}{4}}, \dots, u_{\frac{n+2}{2}}$ . The next vertex  $u_{\frac{n+4}{2}}$  is labeled by 3. The remaining vertices of the cycle are labeled by 1. Then consider the vertices  $v_i$ ,  $1 \leq i \leq n$ . Assign the label 2 to the vertices  $v_i$  where  $1 \leq i \leq \frac{n-2}{4}$  then the vertices  $v_{\frac{n-2}{4}+i}$  where  $1 \leq i \leq \frac{n-2}{4}$  are labeled with 4. Put the labels 1, 3, 1 to the next consecutive vertices  $v_{\frac{n}{2}}, v_{\frac{n+2}{2}}, v_{\frac{n+4}{2}}$ . Finally put the number 3 for the unlabeled vertices.

**Case 4.**  $n \equiv 3 \pmod{4}$ .

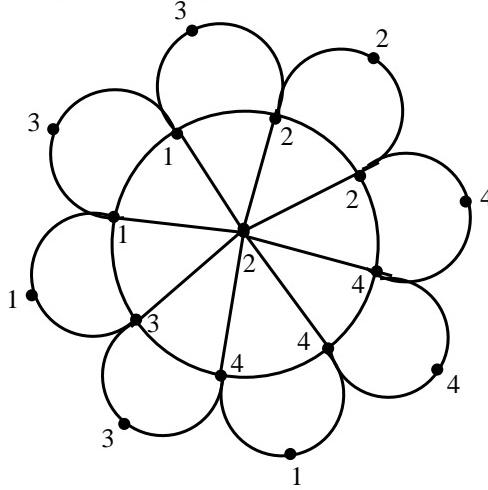
Assign the label 2 to the vertices  $u_i$ , where  $1 \leq i \leq \frac{n+1}{4}$ , then assign 4 to the vertices  $u_{\frac{n+1}{4}+i}$  where  $1 \leq i \leq \frac{n+1}{4}$ . The next vertex  $u_{\frac{n+3}{2}}$  received the label 3. Then assign the label 1 to the remaining vertices of the cycle. Put the integer 2 to the vertex  $u$ . Then we consider the vertices  $v_i$ ,  $1 \leq i \leq n$ . Assign the number 2 to the vertices  $v_1, v_2, \dots, v_{\frac{n-3}{4}}$ . Then assign 4 to the vertices  $v_{\frac{n+1}{4}}, \dots, v_{\frac{n-1}{2}}$ . The next two consecutive vertices  $v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}$  are labeled by 3, 1 respectively. The rest of the unlabeled vertices are labeled by 3.

The Table 2 shows that the above labeling  $f$  is a required 4-prime cordial labeling.  $\square$

Values of $n$	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod{4}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2} + 1$	$2n$	$2n$
$n \equiv 1 \pmod{4}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$2n$	$2n$
$n \equiv 2 \pmod{4}$	$\frac{n}{2}$	$\frac{n}{2} + 1$	$\frac{n}{2}$	$\frac{n}{2}$	$2n$	$2n$
$n \equiv 3 \pmod{4}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$2n$	$2n$

**Table 2**

A 4-prime cordial labeling of  $SF_8$  is given in Figure 1.



**Figure 1**

**Theorem 3.3**  $DH_n$  is 4-prime cordial.

*Proof* Clearly  $DH_n$  consists of  $4n$  vertices and  $6n$  edges. We now give the label to the vertices of  $DH_n$  as follows: Assign the label 2 to the vertices  $u_i$ ,  $1 \leq i \leq n$  and assign the label 4 to the vertices  $v_i$ ,  $1 \leq i \leq n$ . Now we move to the vertices  $y_i$ . Assign the label 1 to the vertices  $y_1, y_2, \dots, y_n$ . Finally assign the label 3 to the vertices  $x_1, x_2, \dots, x_n$ . This vertex labeling  $f$  is obviously a 4-prime cordial labeling of  $DH_n$ . Since,  $v_f(1) = v_f(2) = v_f(3) = v_f(4) = n$  and  $e_f(0) = e_f(1) = 3n$ .  $\square$

**Theorem 3.4** Let  $C_n$  be the cycle  $u_1u_2 \dots u_nu_1$ . Let  $G_n$  be the graph with  $V(G_n) = V(C_n) \cup \{v_i : 1 \leq i \leq n\}$  and  $E(G_n) = E(C_n) \cup \{uv_i : u \in C_n, 1 \leq i \leq n\}$ .

$1 \leq i \leq n\}$  and  $E(G_n) = E(C_n) \cup \{u_i v_i : 1 \leq i \leq n\} \cup \{u_{i+1} v_i : 1 \leq i \leq n-1\}$ . Then  $G_n$  is 4-prime cordial for all  $n \neq 4$ .

*Proof* Clearly for any vertex labeling of  $G_4$ , the maximum possible edges with label 0 is 4. Hence  $G_4$  is not 4-prime cordial.

**Case 1.**  $n \equiv 0 \pmod{4}$ ,  $n \neq 4$ .

Let  $n = 4t$ . Assign the label 2 to the vertices  $u_1, u_2, \dots, u_{2t}$  and 4 to the vertices  $v_1, v_2, \dots, v_{2t}$ . Next assign the label 3 to the vertices  $u_{2t+1}, u_{2t+2}$  and  $u_{2t+3}$ . Next assign the label 1 to the cycle vertices  $u_{2t+4}, u_{2t+6}, \dots, u_{4t}$  and 3 to the vertices  $v_{2t+4}, v_{2t+5}, \dots, v_{4t-1}, v_{4t}$ . Finally assign the remaining non labeled vertices by 1.

**Case 2.**  $n \equiv 1 \pmod{4}$ .

Let  $n = 4t + 1$ . In this case, assign the label 2 to the vertices  $u_1, u_2, \dots, u_{2t+1}$  and 4 to the vertices  $v_1, v_2, \dots, v_{2t+1}$ . Next assign the label 1 to the cycle vertices  $u_{2t+2}, u_{2t+3}, \dots$  and  $u_{4t+1}$ . Finally assign the label 3 to the non labeled vertices  $v_{2t+2}, v_{2t+3}, \dots, v_{4t+1}$ .

**Case 3.**  $n \equiv 2 \pmod{4}$ .

Let  $n = 4t + 2$ . In this case we assign the label 2 to the vertices  $u_i$  ( $1 \leq i \leq 2t + 1$ ) and 4 to the vertices  $v_i$  ( $1 \leq i \leq 2t + 1$ ). Next assign the label 3 to the cycle vertices  $u_{2t+2}, u_{2t+3}$  and  $u_{2t+4}$ . Now we assign the label 1 to the vertices  $u_{2t+5}, u_{2t+6}, \dots, u_{4t+2}$ . Next assign the label 3 to the vertices  $v_{4t+2}, v_{4t+1}, \dots, v_{2t+4}$ . Finally assign the label to the non labeled vertices by 1.

**Case 4.**  $n \equiv 3 \pmod{4}$ .

Let  $n = 4t + 3$ . Assign the label 2 to the vertices  $u_1, u_2, \dots, u_{2t+2}$  and 4 to the vertices  $v_1, v_2, \dots, v_{2t+2}$ . Next assign the label 1 to the vertices  $u_{2t+3}, u_{2t+4}, \dots, u_{4t+3}$ . Finally assign the label to the vertices  $v_{2t+3}, v_{2t+4}, \dots, v_{4t+3}$ .

Hence the vertex labeling given above is obviously a 4-prime cordial labeling of  $G_n$ .  $\square$

### 3.2 Subdivided Graphs

**Theorem 3.5**  $S(K_2 + mK_1)$  is 4-prime cordial.

*Proof* Let  $V(S(K_2 + mK_1)) = \{u, v, w, u_i, v_i, w_i : 1 \leq i \leq m\}$  and  $E(S(K_2 + mK_1)) = \{uv, vw, uu_i, u_i w_i, w_i v_i, v_i u : 1 \leq i \leq m\}$ . Note that  $S(K_2 + mK_1)$  has  $3m + 3$  vertices and  $4m + 2$  edges. The proof is divided into four cases depending upon the nature of  $n$ .

**Case 1.**  $n = 4t$ .

Assign the label 2 to the vertex  $u$ . Next assign the label 2 to the vertices  $u_1, u_2, \dots, u_{3t}$ . We now assign the label 4 to the vertices  $u_{3t+1}, u_{3t+2}, \dots, u_{4t}$ . Next we move to the vertices  $w_i$ . Assign the label 4 to the vertices  $w_1, w_2, \dots, w_{2t}$ . Next assign 3 to the vertices  $w_{3t+1}, w_{3t+2}, \dots, w_{4t}$ . Next assign the label 3 to the vertices  $v_{4t}, v_{4t-1}, \dots, v_{3t+1}$ . We now assign the labels 4, 3 respectively to the vertices  $w, v$ . Finally, assign 1 to all the remaining non labeled vertices.

**Case 2.**  $n = 4t + 1$ .

Assign the labels to the vertices  $u_i, v_i, w_i$ ,  $1 \leq i \leq 4t + 1$ ,  $u, v, w$  as in case 1. Finally assign 2, 4 and 1 respectively to the vertices  $u_{4t+1}, w_{4t+1}$  and  $v_{4t+1}$ .

**Case 3.**  $n = 4t + 2$ .

As in Case 2, assign the labels to the vertices  $u_i, v_i, w_i$ ,  $1 \leq i \leq 4t + 1$ ,  $u, v, w$ . Next assign the labels 2, 1, 3 to the vertices  $u_{4t+2}, w_{4t+2}$  and  $v_{4t+2}$  respectively.

**Case 4.**  $n = 4t + 3$ .

Assign the labels to the vertices  $u_i, v_i, w_i$ ,  $1 \leq i \leq 4t + 2$ ,  $u, v, w$  as in case 3. Finally assign 4, 1, 3 respectively to the vertices  $u_{4t+3}, w_{4t+3}$  and  $v_{4t+3}$ .

Clearly, the above labeling is a 4-prime cordial labeling of  $S(K_2 + mK_1)$ .  $\square$

**Theorem 3.6**  $S(P_n \odot K_1)$  is 4-prime cordial.

*Proof* Let  $P_n$  be the path  $u_1u_2\dots u_n$  and  $V(P_n \odot K_1) = V(P_n) \cup \{v_i : 1 \leq i \leq n\}$ ,  $E(P_n \odot K_1) = E(P_n) \cup \{u_iv_i : 1 \leq i \leq n\}$ . The graph  $S(P_n \odot K_1)$  is obtained by subdividing the edge  $u_iu_{i+1}$  with  $w_i$  and the edge  $u_iv_i$  with  $x_i$ . Note that  $S(P_n \odot K_1)$  has  $4n - 1$  vertices and  $4n - 2$  edges. The proof is divided into four cases.

**Case 1.**  $n \equiv 0 \pmod{4}$ .

Let  $n = 4t$ . Assign the label 2 to the vertices  $u_1, u_2, \dots, u_{4t}$  and 4 to the vertices  $w_1, w_2, \dots, w_{4t-1}$  and  $x_1$ . We now assign the label 1 to the vertices  $x_2, x_3, \dots, x_{4t}$ . Finally assign the label 3 to the pendent vertices  $v_1, v_2, \dots, v_{4t}$ .

**Case 2.**  $n \equiv 1 \pmod{4}$ .

Assign the label to the vertices  $u_i, v_i, x_i$  ( $1 \leq i \leq n - 1$ ) and  $w_i$  ( $1 \leq i \leq n - 2$ ) as in Case 1. Finally assign the labels 2, 4, 3 and 1 respectively to the vertices  $w_{n-1}, u_n, x_n, v_n$ .

**Case 3.**  $n \equiv 2 \pmod{4}$ .

As in Case 2, assign the label to the vertices  $u_i, v_i, x_i$  ( $1 \leq i \leq n - 1$ ) and  $w_i$  ( $1 \leq i \leq n - 2$ ). Next assign the labels 2, 4, 3 and 1 to the vertices  $w_{n-1}, u_n, x_n, v_n$  respectively.

**Case 4.**  $n \equiv 3 \pmod{4}$ .

In this case also assign labels to the vertices except  $w_{n-1}, u_n, x_n, v_n$  as in case 3. Then assign the labels 2, 4, 3 and 1 to the vertices  $w_{n-1}, u_n, x_n, v_n$  respectively.

Clearly, the vertex labeling given in all cases is a 4-prime cordial labeling of  $S(P_n \odot K_1)$ .  $\square$

**Theorem 3.7**  $S(C_3^{(t)})$  is 4-prime cordial.

*Proof* Note that  $S(C_3^{(t)}) \equiv C_6^{(t)}$ . Let the  $i^{th}$  copy of the  $C_6$  be  $u_1^i u_2^i u_3^i u_4^i u_5^i u_6^i$ , where  $1 \leq i \leq t$  and  $u_1^1 = u_1^2 = u_1^3 = u_1^4 = u_1^5 = u_1^6$ .

**Case 1.**  $t$  is even,  $t \geq 4$ .

Assign the label 2 to the central vertex.

**Subcase 1.1**  $t \equiv 0 \pmod{4}$ .

Assign the label 2 to all the vertices of first  $\frac{t}{4}$  copies of the cycle  $C_5$ . Next we move to the  $(\frac{t}{4} + 1)^{th}$  copy. Assign the label 4 to all the vertices of the  $(\frac{t}{4} + 1)^{th}, (\frac{t}{4} + 2)^{th}, \dots, (\frac{t}{2})^{th}$  copies of the cycle  $C_5$ . We now consider the  $(\frac{t}{2} + 1)^{th}$  copy. Assign the label 1 and 3 alternatively to the vertices of the  $(\frac{t}{2} + 1)^{th}$  copy of the cycle. In a similar fashion assign the label 1 and 3 alternatively to the vertices of the  $(\frac{t}{2} + 2)^{th}, \dots, t^{th}$  copy of the cycle  $C_5$ .

**Subcase 1.2**  $t \equiv 2 \pmod{4}$ .

In this case assign the label 2 to all the vertices of first  $\frac{t-2}{4}$  copies of the cycle  $C_5$ . Now consider the  $(\frac{t+2}{4})^{th}$  copy. Assign the label 4 to all the vertices of the cycle  $(\frac{t+2}{4})^{th}$  copy. Similarly assign the label 4 to the  $(\frac{t+6}{4})^{th}, \dots, (\frac{t-2}{2})^{th}$  copies of the cycle  $C_5$ . We now move to the  $(\frac{t}{2})^{th}$  copy. In this copy, assign the label 2 to the vertices  $u_2^t, u_3^t$  and 4 to the vertices  $u_4^t, u_5^t, u_6^t$ . Next assign 1, 3 alternatively to the vertices of the  $(\frac{t}{2} + 1)^{th}, \dots, t^{th}$  copies of the cycle.

**Case 2.**  $t$  is odd.

**Subcase 2.1**  $t \equiv 1 \pmod{4}$ .

As in subcase 1a, assign the label to the vertices of all the  $i^{th}, 1 \leq i \leq t - 1$  copies of  $C_5$ . In the last copy, assign the labels 2, 4, 4, 1 and 3 respectively to the vertices  $u_2^t, u_3^t, u_4^t, u_5^t$  and  $u_6^t$ .

**Subcase 2.2**  $t \equiv 3 \pmod{4}$ .

Assign the label to the vertices of  $(t - 1)$  copies of the cycle as in subcase 1b. Finally, assign the labels 2, 4, 3, 3 and 1 to the vertices  $u_2^t, u_3^t, u_4^t, u_5^t$  and  $u_6^t$  respectively.

The Table 3 establish that the above vertex labeling  $f$  is a 4-prime cordial labeling of  $S(C_3^{(t)})$ ,  $t \geq 4$ .

Nature of t	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$t \equiv 0 \pmod{4}$	$\frac{5t}{4}$	$\frac{5t}{4} + 1$	$\frac{5t}{4}$	$\frac{5t}{4}$	$3t$	$3t$
$t \equiv 1 \pmod{4}$	$\frac{5t-1}{4}$	$\frac{5t+3}{4}$	$\frac{5t-1}{4}$	$\frac{5t+3}{4}$	$3t$	$3t$
$t \equiv 2 \pmod{4}$	$\frac{5t+2}{4}$	$\frac{5t+2}{4}$	$\frac{5t-2}{4}$	$\frac{5t+2}{4}$	$3t$	$3t$
$t \equiv 3 \pmod{4}$	$\frac{5t+1}{4}$	$\frac{5t+1}{4}$	$\frac{5t+1}{4}$	$\frac{5t+1}{4}$	$3t$	$3t$

**Table 3**

$S(C_3^{(1)}) \cong C_6$  is 4-prime cordial follows from Theorem 2.14. The 4-prime cordial labelings of  $S(C_3^{(2)})$  and  $S(C_3^{(3)})$  are given in Figure 2.  $\square$

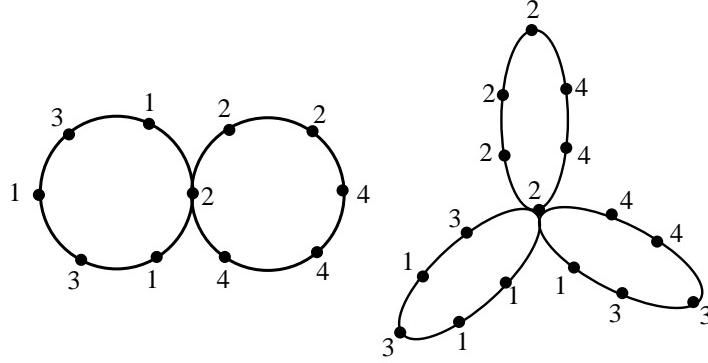


Figure 2

### 3.3 Miscellaneous Graphs

**Theorem 3.8**  $P_n^2$  is 4-prime cordial if and only if  $n \neq 4$ .

*Proof* Let  $P_n$  be the path  $u_1, u_2, \dots, u_n$ . Clearly, the order and size of  $P_n^2$  are  $n$  and  $2n - 3$  respectively. It is easy to verify that  $P_4^2$  does not admit a 4-prime cordial labeling. Let us assume that  $n \neq 4$ .

**Case 1.**  $n \equiv 0 \pmod{4}$ .

Let  $n = 4t$ . Assign the label 1 to the vertices  $u_1, u_3, \dots, u_{2t-1}$ . Next assign the label 3 to the vertices  $u_2, u_4, \dots, u_{2t}$ . Assign the label 2 to the next  $t$  vertices  $u_{2t+1}, \dots, u_{3t}$ . Finally, assign the label 4 to the next  $t$  non-labeled vertices  $u_{3t+1}, \dots, u_{4t}$ .

**Case 2.**  $n \equiv 1 \pmod{4}$ .

**Subcase 2.1**  $n \equiv 1 \pmod{8}$ .

Assign the labels to the vertices of  $u_i$ ,  $1 \leq i \leq n-1$  as in Case 1. Finally, assign the label 2 to the vertex  $u_n$ .

**Subcase 2.1**  $n \equiv 5 \pmod{8}$ .

As in Case 1, assign the labels to the vertices  $u_i$ ,  $1 \leq i \leq n-1$ . Then assign the label 1 to the vertex  $u_n$ .

**Case 3.**  $n \equiv 2 \pmod{4}$ .

**Subcase 3.1**  $n \equiv 2 \pmod{8}$ .

Assign the labels to the vertices of  $u_i$ ,  $1 \leq i \leq n-1$  as in Subcase 2.1. Then assign the label 1 to the vertex  $u_n$ .

**Subcase 3.2**  $n \equiv 6 \pmod{8}$ .

As in Case 1, assign the labels to the vertices  $u_i$ ,  $1 \leq i \leq n-2$ . Then assign the labels 2 and 1 respectively to the vertices  $u_{n-1}$  and  $u_n$ .

**Case 4.**  $n \equiv 3 \pmod{4}$ .

**Subcase 4.1**  $n \equiv 3 \pmod{8}$ .

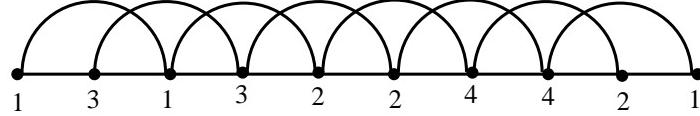
As in Subcase 3.1, assign the labels to the vertices of  $u_i$ ,  $1 \leq i \leq n-1$ . Finally, assign the label 4 to the last vertex  $u_n$ .

**Subcase 4b.**  $n \equiv 7 \pmod{8}$ .

Assign the labels to the vertices  $u_i$ ,  $1 \leq i \leq n-1$  as in Subcase 3.2. Finally, assign the label 4 to the vertex  $u_n$ .

It is easy to verify that the above vertex labeling pattern is 4-prime cordial labeling.  $\square$

The following Figure 3 is an example of a 4-prime cordial labeling of  $P_{10}^2$ .



**Figure 3**

**Theorem 3.9** *The graph  $GL_n$  is 4-prime cordial for  $n > 2$ .*

*Proof* Here we consider the following cases.

**Case 1.**  $n \equiv 0 \pmod{4}$ .

Clearly  $GL_n$  has  $2n$  vertices and  $4n - 3$  edges. Let  $n = 4t$ . Assign the label 2 to the vertices  $u_1, u_2, \dots, u_{2t}$  and assign the label 4 to the vertices  $v_2, v_3, \dots, v_{2t+1}$ . Next assign the label 3, 1 alternatively to the vertices  $u_{2t+1}, u_{2t+2}, \dots, u_{4t}$ . Assign the label 1 to the vertices  $v_{4t}, v_{4t-2}, v_{4t-4}, \dots, v_{2t+5}$  and assign the label 3 to the vertices  $v_{4t-1}, v_{4t-3}, v_{4t-5}, \dots, v_{2t+4}$ . Finally assign the labels 1, 1, 3 and 3 respectively to the vertices  $v_{2t+3}, v_{2t+2}, v_{2t+1}$  and  $v_1$ .

**Case 2.**  $n \equiv 1 \pmod{4}$ .

Assign the label to the vertices  $u_i, v_i$ ,  $1 \leq i \leq n-1$  as in case 1. Next assign the labels 2, 4 to the vertices  $u_n, v_n$  respectively.

**Case 3.**  $n \equiv 2 \pmod{4}$ .

As in Case 2, assign the label to the vertices  $u_i, v_i$ ,  $1 \leq i \leq n-1$ . Next assign the labels 1, 3 respectively to the vertices  $u_n, v_n$ . Finally interchange the labels of  $u_{2t+2}$  and  $u_{2t+3}$ , that is the label of  $u_{2t+2}$  is 3 and the label of  $u_{2t+3}$  is 1.

**Case 4.**  $n \equiv 3 \pmod{4}$ .

As in Case 3, assign the label to the vertices  $u_i, v_i$ ,  $1 \leq i \leq n-1$ . Then assign the labels 2, 4 to the vertices  $u_n, v_n$  respectively. Clearly the above vertex labeling is a 4 prime cordial labeling of  $GL_n$  for all  $n \geq 4$  and  $n \neq 2$ . A 4-prime cordial labeling of  $GL_3$  is shown in the following Figure 4.  $\square$

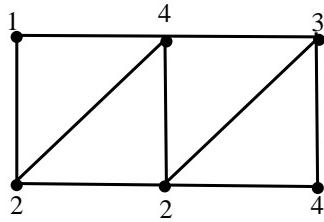


Figure 4

## References

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## Some Results on $\alpha$ -graceful Graphs

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**Abstract:** In this paper we have proved that the graph obtained by merging  $t$  consecutive vertices of two cycles  $C_{4r}$  and  $C_{4s}$  is  $\alpha$ -graceful graph. We also proved that  $G_1$  be an  $\alpha$ -graceful graph and  $G_2$  be a graceful graph joining by path  $P_n$  is graceful in addition we proved  $G_1$  and  $G_2$  be  $\alpha$ -graceful graphs joining by path  $P_n$  is an  $\alpha$ -graceful graph too.

**Key Words:** Cycle, path, graceful labeling,  $\alpha$ -labeling.

**AMS(2010):** 05C78.

### §1. Introduction

A.Rosa [1] defined  $\alpha$ -labeling or ( $\alpha$ -valuation) as a graceful labeling with the additional property that there is an integer  $k$  ( $0 \leq k < E(G)$ ) such that for every  $e = (x, y) \in E(G)$ , either  $f(x) \leq k < f(y)$  or  $f(y) \leq k < f(x)$ . It follows that such a  $k$  must be the smaller of the two vertex labels that yield the edge labeled 1.

In [4] Kaneria et al. proved that the  $\alpha$ -labeling of double path union of some  $\alpha$ -graceful graph like cycle, complete bipartite graph and path. We will consider a simple finite and undirected graph  $G = (V, E)$  on  $|V| = p$  vertices and  $|E| = q$  edges. For a comprehensive bibliography of papers on graph labeling we have refereed Gallian [3]. Here we recall some definitions which are used in this paper.

**Definition 1.1** A function  $f$  is called graceful labeling of a graph  $G = (V, E)$  if  $f : V \rightarrow \{0, 1, \dots, q\}$  is injective and the induced function  $f^* : E \rightarrow \{1, 2, \dots, q\}$  defined as  $f^*(e) = |f(u) - f(v)|$  is bijective for every edge  $e = (u, v) \in E$ . A graph  $G$  is called graceful graph if it admits a graceful labeling.

**Definition 1.2** A function  $f$  is called  $\alpha$ -labeling of a graph  $G = (V, E)$  if  $f$  is a graceful labeling for  $G$  and there exist an integer  $k$  ( $0 \leq k < q - 1$ ) such that for every  $e = (x, y) \in E(G)$ , either  $f(x) \leq k < f(y)$  or  $f(y) \leq k < f(x)$ . A graph  $G$  with an  $\alpha$ -labeling is necessarily bipartite graph. A graph which admits  $\alpha$ -labeling, we call here  $\alpha$ -graceful graph.

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<sup>1</sup>Received July 27, 2016, Accepted May 28, 2017.

## §2. Main Results

**Theorem 2.1** *The graph obtained by merging  $t$  consecutive vertices of two cycle  $C_{4r}$  and  $C_{4s}$  is  $\alpha$ -graceful, where  $t \leq 2[\min\{r, s\}]$  and  $r, s, t \in N$ .*

*Proof* Let  $u_i (i \leq i \leq 4r)$  be consecutive vertices of the cycle  $C_{4r}$  and  $v_1 = u_{4r-t+1}, v_2 = u_{4r-t+2}, \dots, v_t = u_{4r}, v_{t+1}, v_{t+2}, \dots, v_{4s}$  be consecutive vertices of the second cycle  $C_{4s}$ , among the consecutive vertices  $v_1, v_2, \dots, v_t$  of the second cycle are merged with the consecutive vertices  $u_{4r-t+1}, u_{4r-t+2}, \dots, u_{4r}$  of the first cycle  $C_{4r}$  respectively.

Let  $G$  be the graph obtained by merging  $t$  consecutive vertices stated above of the cycle  $C_{4r}$  and  $C_{4s}$ , where  $t \leq 2[\min\{r, s\}]$  and  $r, s, t \in N$ .

The vertex labeling function  $f : V(C_{4r}) \rightarrow \{0, 1, \dots, 4r\}$  is defined by

$$f(u_i) = \begin{cases} 4r - \frac{i-1}{2}, & \forall i = 1, 3, \dots, 4r-1, \\ \frac{i-2}{2}, & \forall i = 2, 4, \dots, 2r, \\ \frac{i}{2}, & \forall i = 2r+2, 2r+4, \dots, 4r \end{cases}$$

is an  $\alpha$ -graceful labeling for the cycle  $C_{4r}$ , where  $k_1 = 2r$ .

Also the vertex labeling function  $g_i : V(C_{4s}) \rightarrow \{0, 1, \dots, 4s\}$  ( $i = 1, 2$ ) is defined by

$$g_1(v_j) = \begin{cases} 4s - \frac{j-1}{2}, & \forall j = 1, 3, \dots, 4s-1, \\ \frac{j-2}{2}, & \forall j = 2, 4, \dots, 2s, \\ \frac{j}{2}, & \forall j = 2s+2, 2s+4, \dots, 4s. \end{cases}$$

$$g_2(v_j) = \begin{cases} \frac{j-1}{2}, & \forall j = 1, 3, \dots, 2s-1, \\ \frac{j+1}{2}, & \forall j = 2s+1, 2s+3, \dots, 4s-1, \\ 4s - \frac{j-2}{2}, & \forall j = 2, 4, \dots, 4s. \end{cases}$$

are  $\alpha$ -graceful labeling for the cycle  $C_{4s}$ , where  $k_2 = 2s$ .

To define a vertex labeling function  $h : V(G) \rightarrow \{0, 1, \dots, q\}$ , where  $q = 4(r+s) - t + 1$ , we take following two cases.

**Case 1.**  $t$  is odd.

Define  $h(u) = f(u)$  if  $f(u) \leq 2r$  and  $f(u) + 4s - t + 1$  if  $f(u) > 2r$  for  $\forall u \in V(C_{4r})$ ,  $h(v) = g_2(v) + 2r - \frac{t-1}{2}$  for  $\forall v \in V(C_{4s}) - V(C_{4r})$ .

**Case 2.**  $t$  is even.

Define  $h(u) = f(u)$  if  $f(u) \leq 2r$  and  $f(u) + 4s - t + 1$  if  $f(u) > 2r$  for  $\forall u \in V(C_{4r})$ ,  $h(v) = g_1(v) + 2r - \frac{t-2}{2}$  for  $\forall v \in V(C_{4s}) - V(C_{4r})$ .

Above defined labeling pattern give the vertex labels  $0, 1, \dots, r-1, r+1, r+2, \dots, 2r, q, q-1, \dots, q+1-2r$  to the vertices of  $C_{4r}$  and the vertex labels  $2r+1, 2r+2, \dots, 2r + \lfloor \frac{2s-t}{2} \rfloor, 2r + \lfloor \frac{2s+4-t}{2} \rfloor, 2r + \lfloor \frac{2s+6-t}{2} \rfloor, \dots, q-2r$  to the vertices of  $C_{4s}$  which are not common vertices of  $C_{4r}$  and  $C_{4s}$ .

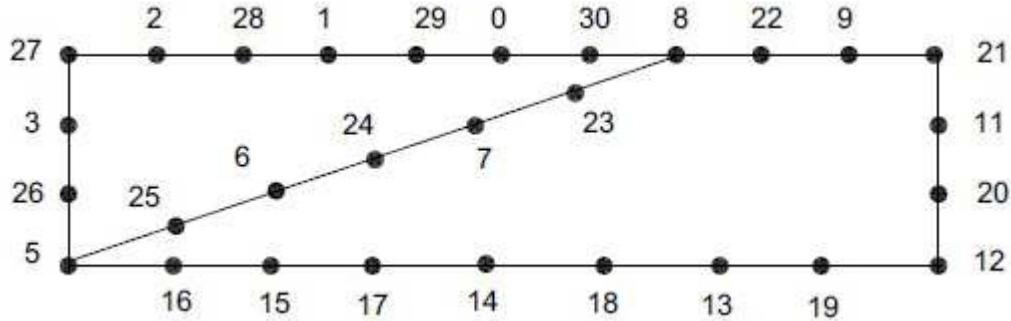
So  $f$  is an injective map. Moreover it produces the edge labels  $q, q-1, \dots, q-4r+1$  to the edges of  $C_{4r}$  and the edge labels  $1, 2, \dots, q-4r$  to the edges of  $C_{4s}$ , which are not common edges of  $C_{4r}$  and  $C_{4s}$ . Thus  $f^*: E(G) \rightarrow \{0, 1, \dots, q\}$  defined by absolute difference of end vertices labels is a bijective map.

It is observed that  $h(v_{4s}) = 2(r+s) + 1 - \frac{t-1}{2}$ ,  $h(v_{4s-1}) = 2(r+s) - \frac{t-1}{2}$  in Case 1 and  $h(v_{4s}) = 2(r+s) - \frac{t-2}{2}$ ,  $h(v_{4s-1}) = 2(r+s) + 1 - \frac{t-2}{2}$  in the Case 2. These produce  $h^*(e = (v_{4s-1}, v_{4s})) = 1$ . Take  $k = \lceil \frac{q}{2} \rceil$ . Now,

$$\begin{aligned} \left\lceil \frac{q}{2} \right\rceil &= \left\lceil \frac{4(r+s) - t + 1}{2} \right\rceil \\ &= 2(r+s) - \left\lfloor \frac{t-1}{2} \right\rfloor \\ &= \begin{cases} 2(r+s) - \frac{t-1}{2} & \text{when } t \text{ is odd} \\ 2(r+s) - \frac{t-2}{2} & \text{when } t \text{ is even} \end{cases} \\ &= \min\{h(v_{4s}), h(v_{4s-1})\}. \end{aligned}$$

Therefore,  $k = \lceil \frac{q}{2} \rceil$  is non-negative integer ( $0 \leq k < q$ ), which satisfied for every  $e = (x, y) \in E(G)$ ,  $\min\{h(x), h(y)\} \leq k < \max\{h(x), h(y)\}$ . Hence,  $G$  admits an  $\alpha$ -graceful labeling  $h$  and so, it is an  $\alpha$ -graceful graph.  $\square$

**Illustration 2.2** Graph obtained by merging 7 consecutive vertices of  $C_{16}$ ,  $C_{20}$  and its  $\alpha$ -graceful labeling are shown in Figure 1.



**Figure 1**  $C_{16} \cup C_{20}$  with seven consecutive common vertices and its  $\alpha$ -labeling, here  $k = 15$ .

**Theorem 2.3** Let  $G_1$  be an  $\alpha$ -graceful graph,  $G_2$  be a graceful graph and  $q_1 = |E(G_1)|$ ,  $q_2 = |E(G_2)|$ . Let  $f_1$  be an  $\alpha$ -graceful labeling for  $G_1$  and  $f_2$  be a graceful labeling for  $G_2$ . Let  $k$  ( $0 \leq k < q_1$ ) be a non-negative integer such that for every  $e_1 = (x, y) \in E(G_1)$ ,  $\min\{f_1(x), f_1(y)\} \leq k < \max\{f_1(x), f_1(y)\}$ . Let  $v \in V(G_1)$  with  $f_1(v) = k$  and  $w \in V(G_2)$  with  $f_2(w) = 0$ . Then the graph obtained by joining  $v$  with  $w$  by a path  $P_n$  is graceful.

*Proof* Take  $V_1 = \{u \in V(G_1) / f_1(u) \leq k\}$  and  $V_2 = V(G_1) - V_1$ . Let  $u_1 = v, u_2, u_3, \dots, u_n = w$  be the vertices of path  $P_n$ . Let  $G$  be the graph obtained by joining  $v$  with  $w$  by a path  $P_n$ .

The vertex labeling function  $g : V(P_n) \rightarrow \{0, 1, \dots, n-1\}$  defined by

$$g(u_i) = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd,} \\ q - \frac{i-2}{2} & \text{if } i \text{ is even, } \forall i = 1, 2, \dots, n \end{cases}$$

is an  $\alpha$ -graceful for the path  $P_n$ , where  $k_1 = \lfloor \frac{n-1}{2} \rfloor$ . Let  $V_3 = \{u \in V(P_n) / f(u) \leq k_1\}$  and  $V_4 = V(P_n) - V_3$ . To define a vertex labeling function  $h : V(G) \rightarrow \{0, 1, \dots, q\}$ , where  $q = q_1 + q_2 + n - 1$ , we take following two cases.

**Case 1.**  $n$  is odd.

Let  $h|_{V_1} = f_1|_{V_1}$ ,  $h|_{V_2} = f_1|_{V_2} + q_2 + n - 1$ ,  $h|_{V_3} = g|_{V_3} + k$ ,  $h|_{V_4} = g|_{V_4} + k + q_2$  and  $h|_{V(G_2)} = f_2|_{V(G_2)} + k + \frac{n-1}{2}$ .

**Case 2.**  $n$  is even.

Let  $h|_{V_1} = f_1|_{V_1}$ ,  $h|_{V_2} = f_1|_{V_2} + q_2 + n - 1$ ,  $h|_{V_3} = g|_{V_3} + k$ ,  $h|_{V_4} = g|_{V_4} + k + q_2$  and  $h|_{V(G_2)} = q_2 + k + \frac{n}{2} - f_2|_{V(G_2)}$ .

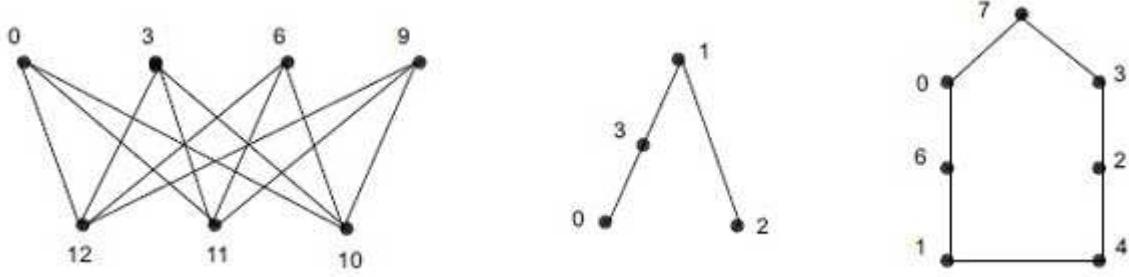
It is observe that  $h(v) = f_1(v) = g(u_1) + k = k$ , which is common vertex of  $G_1$  and  $P_n$ . Also

$$\begin{aligned} h(w) &= \begin{cases} g(w) + k = k + \lfloor \frac{n}{2} \rfloor, & \text{where } n \text{ is odd as } w \in V_3 \\ g(w) + k + q_2 = k + q_2 + \lfloor \frac{n}{2} \rfloor, & \text{where } n \text{ is even as } w \in V_4 \end{cases} \\ &= \begin{cases} k + \lfloor \frac{n}{2} \rfloor + f_2(w), & \text{where } n \text{ is odd as } f_2(w) = 0 \\ k + q_2 + \lfloor \frac{n}{2} \rfloor - f_2(w), & \text{where } n \text{ is even as } f_2(w) = 0 \end{cases}, \end{aligned}$$

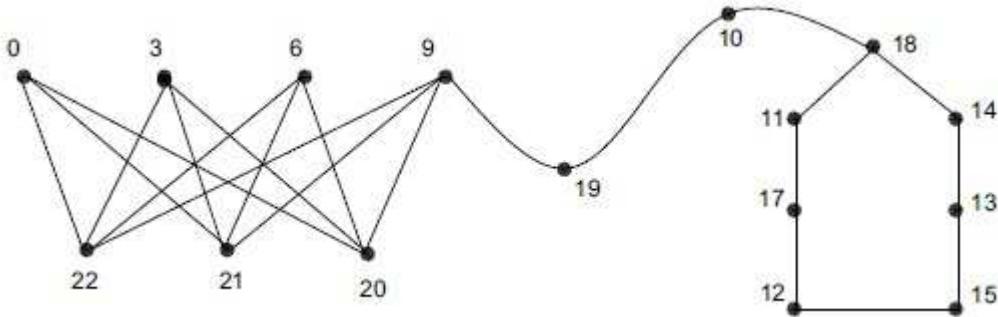
which is common vertex of  $P_n$  and  $G_2$ . Above defined labeling pattern give the vertex labels from  $\{0, 1, \dots, k, q_2 + k + n, \dots, q\}$  to the vertices of  $G_1$ , the vertex labels  $k+1, k+2, \dots, k + \lfloor \frac{n-2}{2} \rfloor, k + q_2 + \lfloor \frac{n+2}{2} \rfloor, \dots, q_2 + k + n - 1$  to the vertices of  $P_n$  except terminal vertices  $v, w$  and the vertex labels from  $\{k + \lfloor \frac{n}{2} \rfloor, k + \lfloor \frac{n}{2} \rfloor + 1, \dots, q_2 + k + \lfloor \frac{n}{2} \rfloor\}$  to the vertices of  $G_2$ .

So,  $h$  is an injective map. Moreover it produced the edge labels  $q, q-1, \dots, q_2+n$  to the graph  $G_1$ , the edge labels  $q_2+1, q_2+2, \dots, q_2+n-1$  to the graph  $P_n$  and the edge labels  $1, 2, \dots, q_2$  to the graph  $G_2$ . Thus  $f^* : E(G) \rightarrow \{0, 1, \dots, q\}$  defined by absolute difference of end vertices labels is a bijective map. Therefore  $G$  admits a graceful labeling  $h$  and so, it is a graceful graph.  $\square$

**Illustration 2.4** The graphs obtained by joining a vertex of an  $\alpha$ -graceful graph  $K_{4,3}$  and a vertex of a graceful graph  $C_7$  by a path  $P_4$  with their require labeling are shown in Figures 2 and 3.



**Figure 2**  $\alpha$ -graceful labeling for  $K_{4,3}$  and  $P_4$ , where  $k = 9, 1$  respectively and graceful labeling for  $C_7$ .



**Figure 3** A graceful labeling for the graph obtained by joining  $K_{4,3}$ ,  $C_7$  by a path  $P_4$ .

**Theorem 2.5** Let  $G_1, G_2$  be  $\alpha$ -graceful graphs and  $q_1 = |E(G_1)|$ ,  $q_2 = |E(G_2)|$ . Let  $f_1, f_2$  be  $\alpha$ -graceful labeling for  $G_1$  and  $G_2$  respectively. Let  $k_1$  ( $0 \leq k_1 < q_1$ ),  $k_2$  ( $0 \leq k_2 < q_2$ ) be two non-negative integers such that for every  $e_i = (x_i, y_i) \in E(G_i)$ ,  $\min\{f_i(x_i), f_i(y_i)\} \leq k_i < \max\{f_i(x_i), f_i(y_i)\}$ ,  $\forall i = 1, 2$ . Let  $v \in V(G_1)$  with  $f_1(v) = k_1$  and  $w \in V(G_2)$  with  $f_2(w) = 0$ . Then the graph obtained by joining  $v$  with  $w$  by a path  $P_n$  is  $\alpha$ -graceful graph.

*Proof* Take  $V_1 = \{u \in V(G_1) / f_1(u) \leq k_1\}$  and  $V_2 = V(G_1) - V_1$ . Let  $u_1 = v, u_2, u_3, \dots, u_n = w$  be the vertices of path  $P_n$ . Let  $G$  be the graph obtained by joining  $v$  with  $w$  by a path  $P_n$ . The vertex labeling function  $g$  on  $P_n$  defined in Theorem 2.3 is an  $\alpha$ -graceful labeling for the path  $P_n$ , where  $k_3 = \lfloor \frac{n-1}{2} \rfloor$ . Let  $V_3 = \{u \in V(P_n) / f(u) \leq k_3\}$  and  $V_4 = V(P_n) - V_3$ .

We define a vertex labeling function  $h : V(G) \rightarrow \{0, 1, \dots, q\}$ , where  $q = q_1 + q_2 + n - 1$  as it defined in Theorem 2.3.

It is observed that  $h(v) = f_1(v) = g(u_1) + k = k$ , which is common vertex of  $G_1$  and  $P_n$ .

Also

$$\begin{aligned} h(w) &= \begin{cases} g(w) + k, & \text{where } n \text{ is odd as } w \in V_3 \\ g(w) + k + q_2, & \text{where } n \text{ is even as } w \in V_4 \end{cases} \\ &= \begin{cases} k + \lfloor \frac{n}{2} \rfloor + f_2(w), & \text{where } n \text{ is odd} \\ k + q_2 + \lfloor \frac{n}{2} \rfloor - f_2(w), & \text{where } n \text{ is even} \end{cases}, \end{aligned}$$

which is common vertex of  $P_n$  and  $G_2$ . Above defined labeling pattern give rise a graceful labeling  $h$  to the graph  $G$ , as discussed in Theorem 2.3.

Since  $f_2$  is an  $\alpha$ -graceful labeling for  $G_2$ , Therefor  $e = (u, v) \in E(G)$  such that  $f_2^*(e) = |f_2(u) - f_2(v)| = 1$ . Science  $\min\{f_2(u), f_2(v)\} \leq k_2 < \max\{f_2(u), f_2(v)\}$  and  $|f_2(u) - f_2(v)| = 1$ , we must say  $\min\{f_2(u), f_2(v)\} = k_2$  and  $\max\{f_2(u), f_2(v)\} = k_2 + 1$ .

Now

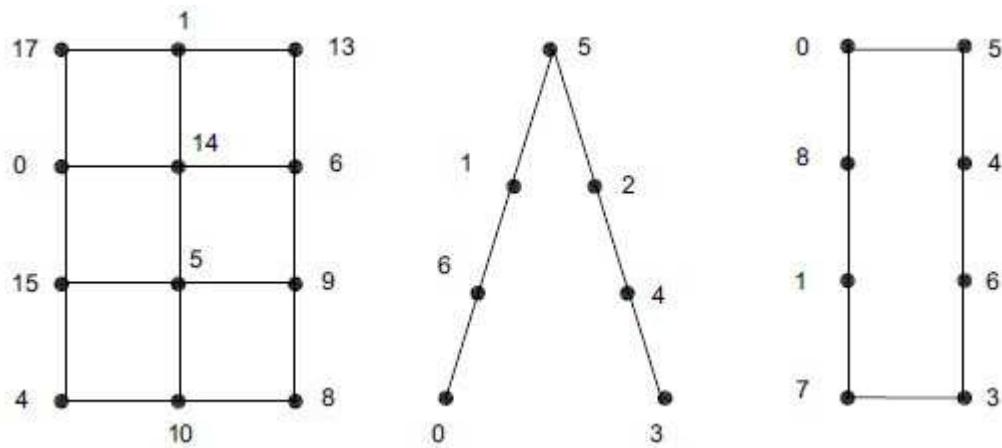
$$\begin{aligned} h^*(e) &= |h(u) - h(v)| = |f_2(u) - f_2(v)| \\ &= f^*(e) = 1. \end{aligned}$$

Take

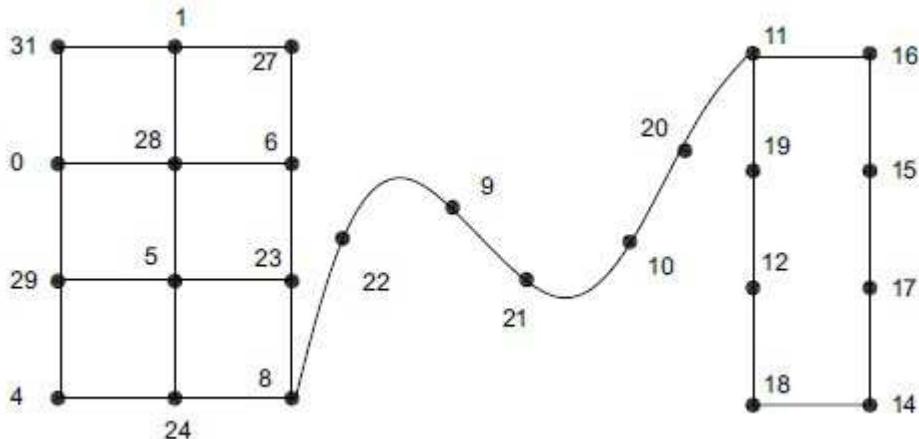
$$\begin{aligned} k &= \min\{h(u), h(v)\} \\ &= \begin{cases} \min\{f_2(u) + k_1 + \lfloor \frac{n}{2} \rfloor, f_2(v) + k_1 + \lfloor \frac{n}{2} \rfloor\}, & \text{where } n \text{ is odd} \\ \min\{q_2 + k_1 + \lfloor \frac{n}{2} \rfloor - f_2(u), q_2 + k_1 + \lfloor \frac{n}{2} \rfloor - f_2(v)\}, & \text{where } n \text{ is even} \end{cases} \\ &= \begin{cases} k_1 + \lfloor \frac{n}{2} \rfloor + \min\{f_2(u), f_2(v)\}, & \text{where } n \text{ is odd} \\ q_2 + k_1 + \lfloor \frac{n}{2} \rfloor - \max\{f_2(u), f_2(v)\}, & \text{where } n \text{ is even} \end{cases} \\ &= \begin{cases} k_1 + k_2 + \lfloor \frac{n}{2} \rfloor, & \text{where } n \text{ is odd} \\ q_2 + k_1 - (k_2 + 1) + \lfloor \frac{n}{2} \rfloor, & \text{where } n \text{ is even} \end{cases} \\ &= \begin{cases} k_1 + k_2 + \frac{n-1}{2}, & \text{where } n \text{ is odd} \\ q_2 - k_2 + k_1 + \frac{n-2}{2}, & \text{where } n \text{ is even} \end{cases}. \end{aligned}$$

Then it is observed that for any  $e = (x, y) \in E(G)$ ,  $\min\{f(x), f(y)\} \leq k < \max\{f(x), f(y)\}$  and so,  $h$  is an  $\alpha$ -graceful labeling for  $G$ . Hence  $G$  is an  $\alpha$ -graceful graph.  $\square$

**Illustration 2.6** The graphs obtained by joining a vertex of an  $\alpha$ -graceful graph  $P_3 \times P_4$  and a vertex of another  $\alpha$ -graceful graph  $C_8$  by a path  $P_5$  and their related labelings are shown in Figures 4 and 5.



**Figure 4**  $\alpha$ -graceful labeling for  $P_3 \times P_4$ ,  $P_7$  and  $C_8$ , where  $k = 8, 3, 4$  respectively.



**Figure 5**  $\alpha$ -graceful labeling for a graph obtained by joining  $P_3 \times P_4$  and  $C_8$  by a path  $P_7$ .

### Acknowledgements

Authors are thankful to the anonymous referee for their valuable comments and suggestions which has added immense value to our research and findings.

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## Supereulerian Locally Semicomplete Multipartite Digraphs

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**Abstract:** A digraph  $D$  is *eulerian* if  $D$  is strongly connected and for every vertex  $v \in V(D)$ ,  $d^+(v) = d^-(v)$ . Bang-Jensen and Thomassé gave the conjecture for a digraph  $D$  that if  $\lambda(D) \geq \alpha(D)$ , then  $D$  is supereulerian. Bang-Jensen and Maddaloni [Journal of Graph Theory, 79(2015)8-20] proved that this conjecture holds for every semicomplete multipartite digraph. In this paper, we generalized the above known results and show that this conjecture holds for every strong locally semicomplete multipartite digraph.

**Key Words:** Digraph, supereulerian digraph, semicomplete digraph, locally semicomplete multipartite digraph.

**AMS(2010):** 05C10, 05C25, 20B25.

### §1. Introduction

A graph  $G$  is *eulerian* if  $G$  is connected without vertices of odd degree, and  $G$  is supereulerian if  $G$  has a spanning eulerian subgraph. The purpose of this paper is to investigate the digraph version of the supereulerian problem. Given a digraph  $D$ , its underling graph, denoted by  $\overline{D}$ , is gotten from  $D$  by overlooking the directions. Let  $A(D)$  and  $V(D)$  be the set of arcs and vertices in  $D$ , respectively. A *walk* in  $D$  is an alternating sequence  $W = x_1e_1x_2e_2 \dots x_{k-1}e_{k-1}x_k$  of vertices  $x_i$  and arcs  $e_j$  from  $D$  such that  $e_i = (x_i, x_{i+1}) \in A(D)$  for  $i = 1, \dots, k-1$ . A walk is *closed* if  $x_1 = x_k$ . If all the arcs of a closed walk are distinct, we call it a *closed trail* (also called a *cycle*). If all the vertices of a walk  $W$  are distinct, we call it an  $(x_1, x_k)$ -*dipath*. A digraph  $D = (V(D), A(D))$  is *strongly connected* if there is a  $(u, v)$ -dipath for any two vertices  $u$  and  $v$ . A digraph  $D$  is *weakly connected* if  $\overline{D}$  is connected. Furthermore,  $D$  is said to be *eulerian* if  $D$  is strongly connected and for every vertex  $v \in V(D)$ ,  $d^+(v) = d^-(v)$ . Equivalently,  $D$  is eulerian if and only if  $D$  itself is a spanning closed trail. A digraph  $D$  is *supereulerian* if  $D$  has a spanning eulerian subdigraph. If  $X$  and  $Y$  are disjoint subsets of  $V(D)$ , let  $A(X, Y) := \{(x, y) \in A(D) : x \in X, y \in Y\}$  (sometime,  $A(X, Y)$  is simply written as  $(X, Y)$ ) and  $d(X, Y) := |(X, Y)|$ . A *circuit*  $C$  of a digraph  $D$  is a connected subdigraph of  $D$  such that

<sup>1</sup>Supported by the National Natural Science Foundation of China (No.11371052).

<sup>2</sup>Received October 12, 2016, Accepted May 29, 2017.

$d_C^+(v) = d_C^-(v) = 1$  for each vertex  $v \in C$ . Let  $[n] = \{1, 2, \dots, n\}$ .

A *semicomplete digraph* is a digraph without nonadjacent vertices. A *locally semicomplete digraph* [3] is a digraph  $D$  that satisfies the following conditions: for every vertex  $x$  of  $D$ , the set  $N^+(x)$  of vertices dominated by  $x$  (respectively, the set  $N^-(x)$  of vertices that dominate  $x$ ) induces a semicomplete digraph. Hence a semicomplete digraph is also locally semicomplete. If each vertex in a semicomplete digraph is blown up, then we get a semicomplete multipartite digraph. A digraph  $D = (V, A)$  is *semicomplete multipartite* if there is a partition  $V_1, V_2, \dots, V_c$  of  $V$  into independent sets so that every vertex in  $V_i$  shares an arc with every vertex in  $V_j$  for  $1 \leq i < j \leq c$ .

Similarly, a *locally semicomplete multipartite digraph*  $D$  is obtained from a locally semicomplete digraph  $F$  with  $V(F) = \{v_1, v_2, \dots, v_c\}$  by blowing up each vertex  $v_i \in V(F)$  into one independent set  $V_i$  in  $D$ , such that  $N_D^{\lambda}(x) = V_{i_1} \cup \dots \cup V_{i_p}$  for any  $x \in V_i$  if and only if  $N_F^{\lambda}(v_i) = \{v_{i_1}, \dots, v_{i_p}\}$ , where  $\lambda \in \{+, -\}$  and  $\{v_{i_1}, \dots, v_{i_p}\} \subseteq V(F)$ . If a digraph  $D = (V, A)$  is locally semicomplete multipartite, then there is a partition  $V_1, V_2, \dots, V_c$  of  $V$  satisfying the following two conditions:

- (1) each  $V_i$ , called a partite set, for  $i \in [c]$  is an independent set;
- (2) the set of vertices dominated by  $x$  (respectively, the set of vertices that dominate  $x$ ) induces a semicomplete multipartite digraph for any vertex  $x \in V$ .

It can be seen that a semicomplete multipartite digraph is locally semicomplete multipartite but the inverse is not right.

Let  $\kappa(D)$  and  $\lambda(D)$  be the vertex-strong connectivity and the arc-strong connectivity of a digraph  $D$ , respectively. While  $\alpha(D)$  and  $\alpha'(D)$  denote the independence number and the matching number of  $D$  respectively, which equal to the independence number and the matching number of a graph  $\overline{D}$ , respectively.

An eulerian factor of  $D$  is a collection of arc-disjoint cycles spanning  $V(D)$ . There are much results about the graph to be supereulerian, for examples to see the surveys [7],[8],[12]. Contrary to the supereulerian properties of graphs, not much work has been done yet for supereulerian digraphs. Gutin [8] discussed supereulerian digraphs by using the connected  $(g, f)$ -factors; Hong et al. [10] and J. Bang-Jensen et al. [4] gave the degree sum conditions for a graph being supereulerian; Bang-Jensen and Thomassé gave the following conjecture in 2011, Bang-Jensen and Maddaloni [4] said the conjecture is unpublished.

**Conjecture 1.1** *Let  $D$  be a digraph. If  $\lambda(D) \geq \alpha(D)$ , then  $D$  is supereulerian.*

Then by using the Hoffman's circulation theorem, the first author and Maddaloni [4] proved that for a digraph  $D$  if  $\lambda(D) \geq \alpha(D)$ , then  $D$  has an eulerian factor and the Conjecture 1.1 hold for all semicomplete multipartite digraphs. Algefari and Lai [2] proved that if  $\lambda(D) \geq \alpha'(D) > 0$ , then  $D$  has a spanning eulerian subdigraph. These two authors, Alsatami and Liu [1] gave the sufficient condition of symmetric digraphs to be supereulerian, and got the result that partially symmetric digraphs are supereulerian.

In this paper, we prove the Conjecture 1.1 for strong locally semicomplete multipartite digraphs.

## §2. Locally Semicomplete Multipartite Digraphs

**Lemma 2.1([4])** *Let  $D$  be a digraph. If  $\lambda(D) \geq \alpha(D)$ , then  $D$  has an Eulerian factor.*

From the definition of a locally semicomplete multipartite digraph, one has the following Lemma 2.2.

**Lemma 2.2** *Let  $D$  be a locally semicomplete multipartite digraph. Let  $x \in V(D)$  and  $v_1, v_2$  be two distinct vertices in  $N_D^+(x)$  (or  $N_D^-(x)$ ), then each of the following results holds.*

- (1) either  $v_1$  and  $v_2$  belong to the same partite set in which there exists no arc between  $v_1$  and  $v_2$  or there exists at least an arc between  $v_1$  and  $v_2$ ;
- (2) If  $v_1$  and  $v_2$  belong to the same partite set, then  $N_D^-(v_1) = N_D^-(v_2)$  and  $N_D^+(v_1) = N_D^+(v_2)$ .

Applying the similar method used in [4] we can prove the following Lemma 2.3.

**Lemma 2.3** *Let  $D$  be a locally semicomplete multipartite digraph. Let  $\mathcal{E}_1$  and  $\mathcal{E}_2$  be two vertex disjoint closed trails. If  $A(\mathcal{E}_1, \mathcal{E}_2) \neq \emptyset$  and  $A(\mathcal{E}_2, \mathcal{E}_1) \neq \emptyset$ , then there exists a closed trail  $\mathcal{E}$  of  $D$  such that  $V(\mathcal{E}) = V(\mathcal{E}_1) \cup V(\mathcal{E}_2)$ .*

*Proof* Consider the bipartite digraph  $B$  with partitions  $V(\mathcal{E}_1)$ ,  $V(\mathcal{E}_2)$  and arcs  $A(\mathcal{E}_1, \mathcal{E}_2) \cup A(\mathcal{E}_2, \mathcal{E}_1)$ . We have two distinct cases.

**Case 1.** Every vertex of  $B$  has positive in- and out-degree. Then, clearly,  $B$  contains a closed trail  $\mathcal{E}$  that connects  $\mathcal{E}_1$  and  $\mathcal{E}_2$ , so  $\mathcal{E} \cup \mathcal{E}_1 \cup \mathcal{E}_2$  is the desired trail.

**Case 2.** There exists one vertex of  $B$  which has no positive in-degree or out-degree.

Let  $v_1v_2\dots v_hv_1$  be a spanning trail of  $\mathcal{E}_1$  and let  $w_1w_2\dots w_kw_1$  be a spanning trail of  $\mathcal{E}_2$ . Without loss of generality, assume that there is a vertex, say  $x$ , of  $\mathcal{E}_1$  such that  $x$  has no out-neighbors in  $\mathcal{E}_2$ . As  $A(\mathcal{E}_1, \mathcal{E}_2) \neq \emptyset$ , there exists another vertex, say  $v_s$  of  $\mathcal{E}_1$  with at least one out-neighbor in  $\mathcal{E}_2$ . Assume  $(v_s, w_t) \in A(\mathcal{E}_1, \mathcal{E}_2)$ .

Now consider the out neighbors  $v_{s+1}$  and  $w_t$  of  $v_s$ . If  $v_{s+1}$  and  $w_t$  are in the same partite set  $V_i$ . By Lemma 2.2(2),  $(w_{t-1}, v_{s+1}) \in A(\mathcal{E}_2, \mathcal{E}_1)$ , thus,  $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \{(w_{t-1}, v_{s+1}), (v_s, w_t)\} \setminus \{(w_{t-1}, w_t), (v_s, v_{s+1})\}$  is the desired trail. Otherwise, by Lemma 2.2(1), there exists an arc between  $w_t$  and  $v_{s+1}$ . If  $(w_t, v_{s+1}) \in A(\mathcal{E}_2, \mathcal{E}_1)$ , then  $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \{(w_t, v_{s+1}), (v_s, w_t)\} \setminus \{(v_s, v_{s+1})\}$  is the desired trail. If  $(w_t, v_{s+1}) \notin A(\mathcal{E}_2, \mathcal{E}_1)$ , then  $(v_{s+1}, w_t) \in A(\mathcal{E}_2, \mathcal{E}_1)$ .

Next we consider two out neighbors  $v_{s+2}$  and  $w_t$  of  $v_{s+1}$ , by the similar discussion as considering  $v_{s+1}$  and  $w_t$ , one has that either  $D$  contains a closed trail  $\mathcal{E}$  such that  $V(\mathcal{E}) = V(\mathcal{E}_1) \cup V(\mathcal{E}_2)$  or there exists an arc  $(v_{s+2}, w_t) \in A(\mathcal{E}_2, \mathcal{E}_1)$ . If the former holds, the theorem is proved. Otherwise,  $(v_{s+2}, w_t) \in A(\mathcal{E}_2, \mathcal{E}_1)$ .

Repeatedly using this procedures, one of the following two results holds: there exists a closed trail  $\mathcal{E}$  of  $D$  such that  $V(\mathcal{E}) = V(\mathcal{E}_1) \cup V(\mathcal{E}_2)$  or every vertex in  $V(\mathcal{E}_1)$  has the out neighbor  $w_t$ . Since  $x$  has no out-neighbors in  $\mathcal{E}_2$ , the result that every vertex in  $V(\mathcal{E}_1)$  has the out neighbor  $w_t$  can not happen, thus, there exists a closed trail  $\mathcal{E}$  of  $D$  such that  $V(\mathcal{E}) = V(\mathcal{E}_1) \cup V(\mathcal{E}_2)$ .  $\square$

By the proof of Lemma 2.3, one has the following Lemma 2.4.

**Lemma 2.4** *Let  $D$  be a locally semicomplete multipartite digraph and  $\mathcal{E}_1, \mathcal{E}_2$  be two vertex disjoint closed trails. Let  $(x, y) \in A(\mathcal{E}_1, \mathcal{E}_2)$  and  $A(\mathcal{E}_2, \mathcal{E}_1) = \emptyset$ , then  $(v, y) \in A(\mathcal{E}_1, \mathcal{E}_2)$  for each  $v \in V(\mathcal{E}_1)$ .*

**Theorem 2.5** *Let  $D$  be a locally semicomplete multipartite digraph. Then  $D$  is supereulerian if and only if it is strong and has an eulerian factor.*

*Proof* If  $D$  is supereulerian, equivalently, there is a spanning eulerian subdigraph in  $D$ , then clearly it is strong and has an eulerian factor which consists of the union of arc disjoint cycles spanning  $V(D)$ .

Now assume that  $D$  is strong and has an eulerian factor  $\mathcal{C}$ , we will show that  $D$  has a spanning eulerian subdigraph. The following is a procedure to produce a spanning eulerian subdigraph of  $D$  for a given eulerian factor.

Form a minimal collection of vertex disjoint closed trails by merging those trails of  $\mathcal{C}$  having common vertices. For any two closed trails  $\mathcal{E}, \mathcal{F}$  in the collection with  $A(\mathcal{E}, \mathcal{F}) \neq \emptyset$  and  $A(\mathcal{F}, \mathcal{E}) \neq \emptyset$ , join  $\mathcal{E}$  and  $\mathcal{F}$  into a closed trail as in Lemma 2.3.

Let  $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_t$  be the collection of closed trails of  $D$  obtained after the first step is no more applicable. Note that all the trails have at least two vertices, and for any two distinct trails  $\mathcal{E}_i$  and  $\mathcal{E}_j$ , at least one of  $A(\mathcal{E}_i, \mathcal{E}_j)$  and  $A(\mathcal{E}_j, \mathcal{E}_i)$  is empty. Let  $D'$  be the digraph with the set of vertices  $\{a_1, \dots, a_t\}$  and the set of arcs  $\{(a_i, a_j) | A(\mathcal{E}_i, \mathcal{E}_j) \neq \emptyset \text{ for } 1 \leq i, j \leq t\}$ . Note that  $D'$  has no 2-circuit. By the fact that  $D$  is strong,  $D'$  is strong. Let  $f : \{a_i, i \in [t]\} \rightarrow \{\mathcal{E}_i, i \in [t]\}$  be a bijective map such that  $f(a_i) = \mathcal{E}_i$  for each  $i \in [t]$ . We has the following claim.

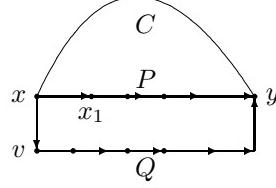
**Claim 1.**  $D'$  has a hamiltonian (directed) circuit.

**Proof of Claim 1.** Let  $C$  be a circuit of  $D'$  with the maximum number of vertices. If  $V(D') = V(C)$ , then  $C$  is our desired circuit. So suppose by contradiction that  $|V(C)| < t$ . Since  $D'$  is strong, there exists a dipath  $Q$  with only two ends in  $C$  and  $Q$  has at least three vertices. Let  $Q$  be chosen such that the length of the shortest dipath  $P$  in  $C$  between the endpoints of  $Q$  is minimum. Let  $x, v, y$  be the first, second, and last vertex of  $Q$  (to see Fig.1). Let  $P = \langle x, x_1, \dots, y \rangle$  and  $Q = \langle x, v, \dots, y \rangle$ .

Since  $C$  is a circuit with the maximum length,  $y$  can not be  $x_1$  nor a vertex  $x$ . In deed, if  $y = x_1$ , then circuit, gotten from  $C$  by replacing  $P$  with  $Q$ , has larger length than that of  $C$  which is a contradiction.

If  $y = x$ , the out-neighbor of  $x$  in  $C$  is denoted by  $x_1$ . We will show that  $(x_1, v) \in A(D')$ . In fact, we consider  $v$  and  $x_1$ . Since  $(x, v), (x, x_1) \in D'$  and  $D'$  has no 2-circuits, by Lemma ??, there exist  $\varepsilon_1 \in V(f(x)), \varepsilon_2 \in V(f(x_1))$  and  $\varepsilon_3 \in V(f(v))$  such that  $(\varepsilon_1, \varepsilon_2), (\varepsilon_1, \varepsilon_3) \in A(D)$ . By the definition of the locally semicomplete multipartite digraph, either  $\varepsilon_2$  and  $\varepsilon_3$  belong to the same partite set or there exists an arc between  $\varepsilon_2$  and  $\varepsilon_3$  in  $D$ . If  $\varepsilon_2$  and  $\varepsilon_3$  belong to the same partite set in  $D$ . Since  $f(v)$ , as a closed trail of  $D$ , has at least two vertices and there exists a vertex, say  $\varepsilon_4 \in V(f(v))$  such that  $(\varepsilon_4, \varepsilon_3) \in A(D)$ . By Lemma 2.2(2),  $(\varepsilon_4, \varepsilon_2) \in A(D)$  which implies that  $(v, x_1) \in A(D')$ , then  $C_1 := C \cup \{(x, v), (v, x_1)\} - \{(x, x_1)\}$  is the circuit of  $D'$  and  $|C_1| \geq |C|$  which contradicts the choice of  $C$ . Now assume that there exists an arc between

$\varepsilon_2$  and  $\varepsilon_3$  in  $D$ . If  $(\varepsilon_3, \varepsilon_2) \in A(D)$ , it implies that  $(v, x_1) \in A(D')$  which is a contradiction. So  $(\varepsilon_2, \varepsilon_3) \in A(D)$ , it implies that  $(x_1, v) \in A(D')$ .



**Fig.1** Circuit  $C$  with the dipath  $Q$

By the same reason as considering  $v$  and  $x_1$ , then consider  $v$  and the out-neighbor, say  $x_2$ , of  $x_1$  in  $C \cap D'$ . We get that  $(x_2, v) \in A(D')$ . Repeatedly using this procedures, we can derive that  $(t, v) \in A(D')$  for every  $t \in V(C)$ . Let  $z$  be the in-neighbor of  $x$  in  $C$ . Note that  $y = x$ . By  $(z, v) \in A(D')$ , then  $C_1 := C \cup Q \cup \{(z, v)\} - \{(z, x), (x, v)\}$  is the circuit of  $D'$  with  $|C_1| \geq |C|$  which contradicts the choice of  $C$ . As a result,  $y$  can not equal  $x_1$  nor a vertex  $x$ .

Since  $(x, v)$  and  $(x, x_1)$  are distinct arcs of  $D'$ , there exists  $t_1, t_2 \in V(f(x))$ ,  $z_1 \in V(f(v))$  and  $z_2 \in V(f(x_1))$  such that  $(t_1, z_1) \in A(f(x), f(v))$  and  $(t_2, z_2) \in A(f(x), f(x_1))$  in  $D$ . By Lemma 2.4 and no 2-circuit in  $D'$ ,  $(t_1, z_2) \in A(f(x), f(x_1))$  in  $D$ . Since  $z_1$  and  $z_2$  are the out-neighbors of  $t_1$  in  $D$ . By Lemma 2.2, either  $z_1$  and  $z_2$  are in the same partite set or there exists an arc between  $z_1$  and  $z_2$  in  $D$ . The later case can not happen. In fact, the arc between  $z_1$  and  $z_2$  in  $D$  implies that there exists an arc between  $v$  and  $x_1$  in  $D'$ . If  $(v, x_1) \in D'$ , then  $C \cup \{(x, v), (v, x_1)\} - \{(x, x_1)\}$  is the circuit with the length longer than that of  $C$ , it contradicts with the choice of  $C$ . If  $(x_1, v) \in D'$ , then it contradicts with the shortest dipath  $P$  in  $C$ . As a result, there exists no arc between  $v$  and  $x_1$  in  $D'$ . Now assume that  $z_1$  and  $z_2$  are in the same partite set in  $D$ , since  $f(v)$  contains at least two vertices and  $f(v)$  is a closed trail of  $D$ , there exists  $z_3 \in f(v)$  such that  $(z_1, z_3) \in A(f(v)) \subseteq D$ , by Lemma 2.2(2),  $(z_2, z_3) \in A(D)$ , it implies that  $(x_1, v) \in A(D')$  which is a contradiction. The proof of Claim is finished.

By Claim 1,  $D'$  has a hamiltonian circuit  $C$ . Without loss of generality, let  $C = a_1a_2 \cdots a_ta_1$ . By Lemma 2.4 and no 2-circuits in  $D'$ , there exists  $w_i \in \mathcal{E}_i$  for each  $i \in [t]$  such that  $(v_{i-1}, w_i) \in A(\mathcal{E}_{i-1}, \mathcal{E}_i)$  for each  $v_{i-1} \in V(\mathcal{E}_{i-1})$  and  $2 \leq i \leq t$ , and  $(v_t, w_1) \in A(\mathcal{E}_t, \mathcal{E}_1)$  for each  $v_t \in V(\mathcal{E}_t)$ . As a result,  $w_1w_2 \cdots w_tw_1$  is a circuit of  $D$ . Then  $D$  is supereulerian with a eulerian digraph  $(\bigcup_{i=1}^t \mathcal{E}_i) \cup \{w_1w_2 \cdots w_tw_1\}$ .  $\square$

From Theorem 2.5 and Lemma 2.1, we can derive the following Theorem 2.6.

**Theorem 2.6** *Let  $D$  be a strong locally semicomplete multipartite digraph. If  $\lambda(D) \geq \alpha(D)$ , then  $D$  is supereulerian.*

Since a semicomplete multipartite digraph is the strong locally semicomplete multipartite digraph, thus the following known result can be gotten directly.

**Corollary 2.7([4])** *Let  $D$  be a semicomplete multipartite digraph. If  $\lambda(D) \geq \alpha(D)$ , then  $D$  is supereulerian.*

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## Non-Existence of Skolem Mean Labeling for Five Star

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**Abstract:** In this paper, we prove if  $\ell \leq m < n$ , the five star  $G = K_{1,\ell} \cup K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$  is not a skolem mean graph if  $|m-n| > 4 + 3\ell$  for  $\ell = 2, 3, \dots$  and  $m = 2, 3, \dots$ .

**Key Words:** Labeling, Smarandachely edge  $m$ -labeling  $f_S^*$ , skolem mean labeling, skolem mean graph, star.

**AMS(2010):** 05C78.

### §1. Introduction

Let  $G$  be a simple graph. A vertex labeling of  $G$  is an assignment  $f : V(G) \rightarrow \{1, 2, 3, \dots, p+q\}$  be an injection. For a vertex labeling  $f$ , the induced Smarandachely edge  $m$ -labeling  $f_S^*$  for an edge  $e = uv$ , an integer  $m \geq 2$  is defined by  $f_S^*(e) = \left\lceil \frac{f(u) + f(v)}{m} \right\rceil$ . Then  $f$  is called a Smarandachely super  $m$ -mean labeling if  $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \dots, p+q\}$ . Particularly, in the case of  $m = 2$ , we know that

$$f^*(e) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even;} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

Such a labeling is usually called a mean labeling. A graph that admits a Smarandachely super mean  $m$ -labeling is called a Smarandachely super  $m$ -mean graph, particularly, a skolem mean graph if  $m = 2$

In [2], we proved the following theorems to study the existence of skolem mean graphs. We proved the three star  $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$  is a skolem mean graph if  $|m-n| = 4 + \ell$  for  $\ell = 1, 2, 3, \dots$ ;  $m = 1, 2, 3, \dots$  and  $\ell \leq m < n$ . The three star  $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$  is not a skolem mean graph if  $|m-n| > 4 + \ell$  for  $\ell = 1, 2, 3, \dots$ ;  $m = 1, 2, 3, \dots$  and  $\ell \leq m < n$ . The four star  $K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$  is a skolem mean graph if  $|m-n| = 4 + 2\ell$  for  $\ell = 2, 3, \dots$ ;  $m = 2, 3, \dots$  and  $\ell \leq m < n$ . The four star  $K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$  is not a skolem mean graph if  $|m-n| > 4 + 2\ell$  for  $\ell = 2, 3, \dots$ ;  $m = 2, 3, \dots$  and  $\ell \leq m < n$ . In [3]. The five star  $K_{1,\ell} \cup K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$  is a skolem mean graph if  $|m-n| = 4 + 3\ell$  for  $\ell = 2, 3, \dots$ ;

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<sup>1</sup>Received August 23, 2016, Accepted May 30, 2017.

$m = 2, 3, \dots$  and  $\ell \leq m < n$ . Further, we prove the four star  $K_{1,1} \cup K_{1,1} \cup K_{1,m} \cup K_{1,n}$  is a skolem mean graph if  $|m-n| = 7$  for  $m = 1, 2, 3, \dots$  and  $1 \leq m < n$ ; The four star  $K_{1,1} \cup K_{1,1} \cup K_{1,m} \cup K_{1,n}$  is not a skolem mean graph if  $|m-n| > 7$  for  $m = 1, 2, 3, \dots$  and  $1 \leq m < n$ ; The five star  $K_{1,1} \cup K_{1,1} \cup K_{1,1} \cup K_{1,m} \cup K_{1,n}$  is a skolem mean graph if  $|m-n| = 8$  for  $m = 1, 2, 3, \dots$  and  $1 \leq m < n$ .

**Definition 1.1** *The five star is the disjoint union of  $K_{1,a}, K_{1,b}, K_{1,c}, K_{1,d}$  and  $K_{1,e}$  and is denoted by  $K_{1,a} \cup K_{1,b} \cup K_{1,c} \cup K_{1,d} \cup K_{1,e}$ .*

## §2. Main Result

**Theorem 2.1** *The five star  $G = K_{1,\ell} \cup K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$  is not a skolem mean graph if  $|m-n| > 4 + 3\ell$  for  $\ell = 2, 3, \dots$  and  $m = 2, 3, \dots$*

*Proof* Let  $G = 4K_{1,2} \cup K_{1,13}$ , where  $V(G) = \{v_{i,j} : 1 \leq i \leq 4; 0 \leq j \leq 2\} \cup \{v_{5,j} : 0 \leq j \leq 13\}$  and  $E(G) = \{v_{i,0} : v_{i,j} : 1 \leq i \leq 4; 1 \leq j \leq 2\} \cup \{v_{5,0}v_{5,j} : 1 \leq j \leq 13\}$ . Then,  $p = 26$  and  $q = 21$ . Suppose  $G$  is a skolem mean graph. Then there exists a function  $f$  from the vertex set of  $G$  to  $\{1, 2, 3, \dots, p\}$  such that the induced map  $f^*$  from the edge set of  $G$  to  $\{2, 3, 4, \dots, p\}$  defined by

$$f^*(e = uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{iff } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{iff } f(u) + f(v) \text{ is odd} \end{cases}$$

then the resulting edges get distinct labels from the set  $\{2, 3, \dots, p\}$ .

Let  $t_{i,j}$  be the label given to the vertex  $v_{i,j}$  for  $1 \leq i \leq 4; 0 \leq j \leq 2$  and  $v_{5,j}$  for  $0 \leq j \leq 13$  and  $X_{i,j}$  be the corresponding edge label of the edge  $v_{i,0}v_{i,j}$  for  $1 \leq i \leq 5; 0 \leq j \leq 2$  and  $v_{5,0}v_{5,j}$  for  $1 \leq j \leq 13$ .

Let us first consider the case that  $t_{5,0} = 26$ . If  $v_{5,j} = 2n$  and  $t_{5,k} = 2n+1$  for some  $n$  and for some  $j$  and  $k$  then  $f^*(v_{5,0}v_{5,j}) = \frac{26+2n}{2} = 13+n = f^*(v_{5,0}v_{5,k})$ . This is not possible as  $f^*$  is a bijection.

Therefore the thirteen vertices  $t_{5,j}$  for  $1 \leq j \leq 13$  are among the 13 numbers (1 or 2), (3 or 4), (5 or 6), (7 or 8), (9 or 10), (11 or 12), (13 or 14), (15 or 16), (17 or 18), (19 or 20), (21 or 22), (23 or 24) and 25.

Primarily,  $t_{5,2}$  is either of 23 or 24. We first consider the case that  $t_{5,2} = 23$ .

**Case 1.**  $t_{5,2} = 23$ .

We have  $t_{5,0} = 26; t_{5,1} = 25; t_{5,2} = 23; t_{1,0} = 24$ . Now 24 is a label of either  $t_{i,0}$  for  $1 \leq i \leq 4$  or  $t_{i,j}$  for  $1 \leq i \leq 4; 1 \leq j \leq 2$ . That is 24 is a label of pendent or non pendent vertex in a  $k_{1,2}$  component of  $G$ . Let us assume that  $t_{1,0} = 24$ .

**Subcase 1.1**  $t_{1,0} = 24$ .

We have  $t_{5,0} = 26; t_{5,1} = 25; t_{5,2} = 23; t_{1,0} = 24$ . If  $t_{1,0} = 24$  then  $t_{1,1}$  take the values 1 or 2. As  $t_{1,1} \geq 3$  would imply that  $X_{1,1} \geq 14$  this is not possible. The corresponding edge labels are  $X_{1,1} = 13$ .

Next  $t_{5,3}$  is either 21 or 22. If  $t_{5,3} = 21$  then  $t_{2,0} = 22$ . If  $t_{2,1} = 3$  or 4 then  $X_{2,1} = \frac{22+3 \text{ or } 4}{2} = 13$  this is not possible.

Similarly, if  $t_{5,3} = 22$  then  $t_{2,0} = 21$ . Then  $t_{2,1}$  take the value 3 or 4. The corresponding edge labels are  $X_{2,1} = 12$ ,  $X_{1,1} = 13$ .

If  $t_{2,2} \geq 5$  then  $X_{2,2} \geq 14$  this is not possible. Hence it is not possible that  $t_{1,0} = 24$ . That is 24 is not a label of a non-pendent vertex in  $k_{1,2}$  component of G. Next we consider the case that 24 is a label of a pendent vertex in a  $k_{1,2}$  component of G. Let us assume that  $t_{1,1} = 24$ .

**Subcase 1.2**  $t_{1,1} = 24$ .

We have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 23$ ;  $t_{1,1} = 24$ . If  $t_{1,0} \geq 3$  then  $X_{1,1} \geq 14$ . This is not possible. Hence the value of  $t_{1,0}$  is 1 or 2.

First,  $t_{1,0} = 1$  or 2. We have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 23$ ;  $t_{1,1} = 24$ . Then  $X_{1,1} = 13$ . Now  $t_{5,3}$  is either of 21 or 22.

Next case let,  $t_{5,3} = 21$  and hence  $t_{2,1} = 22$ . If  $t_{2,0} \geq 5$  then  $X_{2,1} \geq 14$ . This is not possible. If  $t_{2,0} = 3 \text{ or } 4$  then  $X_{2,1} = \left(\frac{26+3 \text{ or } 4}{2}\right) = 15$  this is not possible.

Suppose  $t_{5,3} = 22$  and hence  $t_{2,1} = 21$ . We have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 23$ ;  $t_{1,1} = 24$ ;  $t_{1,0} = 1$  or 2;  $t_{2,1} = 21$ ;  $t_{2,0} = 3$ . Then  $X_{1,1} = 13$ ,  $X_{2,1} = 12$ . Now  $t_{5,4}$  is either of 19 or 20.

Consider the case that  $t_{5,4} = 19$  hence  $t_{3,1} = 20$ . We have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 23$ ;  $t_{1,1} = 24$ ;  $t_{5,3} = 22$ ;  $t_{2,1} = 21$ ;  $t_{2,0} = 3$ . Here the value  $t_{3,0} \geq 4$  then  $X_{3,1} \geq 13$  this is not possible. If  $t_{5,4} = 20$ , then  $t_{3,1} = 19$ . Notice that  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 23$ ;  $t_{1,1} = 24$ ;  $t_{5,3} = 22$ ;  $t_{2,1} = 21$ ;  $t_{2,0} = 3$ . Here the value  $t_{3,0} \geq 4$  then  $X_{3,1} \geq 12$ . This is not possible. Hence  $t_{5,4} \neq 19$ .

Similarly  $t_{5,4} \neq 20$ ;  $t_{5,3} \neq 22$ ;  $t_{5,3} \neq 21$ . Hence  $t_{1,0} \neq 1 \text{ or } 2$  therefore  $t_{1,1} \neq 24$ ;  $t_{5,2} \neq 23$ .

**Case 2.**  $t_{5,2} = 24$ .

Now 23 is a label of either  $t_{i,0}$  for  $1 \leq i \leq 4$  or  $t_{i,j}$  for  $1 \leq i \leq 4$ ;  $1 \leq j \leq 2$ ; that is 23 is a label of pendent or non-pendent vertex in a  $K_{1,2}$  component of G.

**Subcase 2.1**  $t_{1,0} = 23$ .

We have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{1,0} = 23$ . If  $t_{1,0} = 23$  then  $t_{1,1}$  and  $t_{1,2}$  take the values of 1 and 2 or 3 as  $t_{1,1} \geq 4$  would imply that  $X_{1,1} \geq 14$  is not possible. The corresponding edge labels are  $X_{1,1} = 12$ ;  $X_{1,2} = 13$ .

Now  $t_{5,3}$  is either of 21 or 22. If  $t_{5,3} = 21$  then  $t_{2,0} = 22$  then  $X_{5,3} = \frac{26+21}{2} = 24$  and  $t_{2,j} \geq 4$  and this is not possible. As  $t_{2,j} \geq 4$  would imply that  $X_{2,j} \geq 13$  and this not possible.

Similarly  $t_{5,3} = 22$  then  $X_{5,3} = \frac{26+22}{2} = 24$ ;  $t_{2,0} = 21$  and also  $t_{2,j} \geq 4$  this is not possible. As  $t_{2,j} \geq 4$  would imply that  $X_{2,j} \geq 13$  and this not possible.

Hence, it is not possible that  $t_{1,0} = 23$  that is 23 is not a label of non-pendent vertex in  $K_{1,2}$  component of G.

Next we consider the case that 23 is a label of a pendent vertex in a  $K_{1,2}$  component of G. Let us assume that  $t_{1,1} = 23$ .

**Subcase 2.2**  $t_{1,1} = 23$ .

We have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{1,1} = 23$ . If  $t_{1,0} \geq 4$  then  $X_{1,1} \geq 14$  and this is not possible. Hence the value of  $t_{1,0}$  can either be 1 or 2 or 3. There exist two cases, i.e.,  $t_{1,0} = 1$  and  $t_{1,0} = 2$  or 3.

**Subcase 2.2.1**  $t_{1,0} = 1$ .

We have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{1,0} = 1$ ;  $t_{1,1} = 23$ . Then  $X_{1,1} = 12$ . Now  $t_{5,3}$  is either of 21 or 22.

Let  $t_{5,3} = 21$  hence  $t_{2,1} = 22$ . If  $t_{2,0} \geq 5$  then  $X_{2,1} \geq 14$  and is not possible. If  $t_{2,0} = 2$  then  $X_{2,1} = \frac{26+2}{2} = 12$  and this is not possible. Hence  $t_{2,0}$  is either of 3 or 4. We have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{1,0} = 1$ ;  $t_{5,3} = 21$ ;  $t_{2,1} = 22$ ;  $t_{2,0} = 3$  or 4 then  $X_{1,1} = 12$ ;  $X_{2,1} = 13$ .

Now  $t_{5,4}$  is either 19 or 20. Assume  $t_{5,4} = 19$  Hence  $t_{3,1} = 20$ . If  $t_{3,0} \geq 5$  then  $X_{3,1} \geq 13$  and is not possible. Hence  $t_{3,0}$  is 2. Notice that  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{5,3} = 21$ ;  $t_{1,0} = 1$ ;  $t_{2,1} = 22$ ;  $t_{2,0} = 3$  or 4;  $t_{3,1} = 20$ ;  $t_{3,0} = 2$  then  $X_{1,1} = 12$ ;  $X_{2,1} = 13$ ;  $X_{3,1} = 11$ .

Now  $t_{5,5}$  is either 17 or 18. Consider  $t_{5,5} = 17$ . Hence  $t_{4,1} = 18$ . We have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{5,3} = 21$ ;  $t_{5,4} = 19$ ;  $t_{1,0} = 1$ ;  $t_{1,1} = 23$ ;  $t_{2,1} = 22$ ;  $t_{2,0} = 3$  or 4;  $t_{3,0} = 2$ ;  $t_{3,1} = 20$ ;  $t_{4,1} = 18$ . Here the value  $t_{4,0} \geq 5$  then  $X_{4,1} \geq 12$ , which is not possible.

Let  $t_{5,5} = 18$  and hence  $t_{4,1} = 17$ . We have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{5,3} = 21$ ;  $t_{5,4} = 19$ ;  $t_{5,5} = 18$ ;  $t_{1,0} = 1$ ;  $t_{1,1} = 23$ ;  $t_{2,1} = 22$ ;  $t_{2,0} = 3$  or 4;  $t_{3,0} = 2$ ;  $t_{3,1} = 20$ ;  $t_{4,1} = 17$ .

If the value  $t_{4,0} \geq 5$  then  $X_{4,1} \geq 11$ , which is not possible. Hence  $t_{5,4} \neq 19$ . Similarly we can prove  $t_{5,4} \neq 20$  and therefore  $t_{5,3} \neq 21$ .

Consider the case that  $t_{5,3} = 22$  and hence  $t_{2,1} = 21$ . If  $t_{2,0} \geq 6$  then  $X_{2,1} \geq 14$  and is not possible. Hence the value of  $t_{2,0}$  can either of 4 or 5.

First we consider  $t_{2,0} = 4$  or 5. We have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{5,3} = 22$ ;  $t_{1,1} = 23$ ;  $t_{1,0} = 1$ ;  $t_{2,1} = 21$ ;  $t_{2,0} = 4$  or 5, then  $X_{1,1} = 12$  and  $X_{2,1} = 13$ .

Now  $t_{5,4}$  is either 19 or 20. Considering,  $t_{5,4} = 19$  and  $t_{3,1} = 20$ . If  $t_{3,0} \geq 7$  then  $X_{3,1} \geq 14$  and is not possible. Hence the value of  $t_{3,0}$  can either be 2 or 6.

If  $t_{3,0} = 6$  then  $X_{3,1} = \frac{20+6}{2} = 13$ , which is not possible. Hence  $t_{3,0}$  is 2. Notice that  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{5,3} = 22$ ;  $t_{1,1} = 23$ ;  $t_{1,0} = 1$ ;  $t_{2,1} = 21$ ;  $t_{2,0} = 4$  or 5;  $t_{3,1} = 20$ ;  $t_{3,0} = 2$ .

Now  $t_{5,5}$  is either 17 or 18. Let us consider  $t_{5,5} = 17$  and  $t_{4,1} = 18$ . Notice that  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{5,3} = 22$ ;  $t_{1,1} = 23$ ;  $t_{1,0} = 1$ ;  $t_{2,1} = 21$ ;  $t_{2,0} = 4$  or 5;  $t_{3,1} = 20$ ;  $t_{3,0} = 2$ ;  $t_{4,1} = 18$ .

Here the value  $t_{4,0} = 3$  then  $X_{4,1} = \frac{18+3}{2} = 11$ , which is not possible.

Now  $t_{5,5}$  is either 18 or 17. Let  $t_{5,5} = 18$  and  $t_{4,1} = 17$ . Notice that  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{5,3} = 22$ ;  $t_{1,1} = 23$ ;  $t_{1,0} = 1$ ;  $t_{2,1} = 21$ ;  $t_{2,0} = 4$  or 5;  $t_{3,1} = 20$ ;  $t_{3,0} = 2$ ;  $t_{4,1} = 17$ ;  $t_{4,0} = 3$ .

Now  $t_{5,6}$  is either 15 or 16. If  $t_{5,6} = 15$  and  $t_{5,1} = 16$ , we have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{5,3} = 22$ ;  $t_{1,1} = 23$ ;  $t_{1,0} = 1$ ;  $t_{2,1} = 21$ ;  $t_{2,0} = 4$  or 5;  $t_{3,1} = 20$ ;  $t_{3,0} = 2$ ;  $t_{4,1} = 17$ ;  $t_{4,0} = 3$ ;  $t_{5,1} = 16$ . Here the value of  $t_{5,0} \geq 6$ . This is not possible.

If  $t_{5,6} = 16$  and  $t_{5,1} = 15$ , we have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{5,3} = 22$ ;  $t_{1,1} = 23$ ;  $t_{1,0} = 1$ ;  $t_{2,1} = 21$ ;  $t_{2,0} = 4$  or 5;  $t_{3,1} = 20$ ;  $t_{3,0} = 2$ ;  $t_{4,1} = 17$ ;  $t_{4,0} = 3$ ;  $t_{5,1} = 15$ . Here the value of  $t_{5,0} \geq 6$ . This is not possible. Hence  $t_{5,4} \neq 19$ .

Similarly  $t_{5,4} \neq 20$  and  $t_{2,0} \neq 4$  or  $5$ . Therefore  $t_{5,3} \neq 18$ . Hence  $t_{1,0} \neq 1$ .

**Subcase 2.2.2**  $t_{1,0} = 2$  or  $3$ .

In this case, we have  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{1,1} = 23$ . Then  $X_{1,1} = 13$ .

Now  $t_{5,3}$  is either  $21$  or  $22$ . If  $t_{5,3} = 21$  and  $t_{2,1} = 22$ . If  $t_{2,0} \geq 4$  then  $X_{2,1} \geq 13$ . This is not possible. Hence  $t_{2,0}$  is  $1$ . Notice that  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{5,3} = 21$ ;  $t_{1,1} = 23$ ;  $t_{1,0} = 2$  or  $3$ ;  $t_{2,0} = 1$ .

Now  $t_{5,4}$  is either  $19$  or  $20$ . Suppose  $t_{5,4} = 19$  and  $t_{3,1} = 20$ . Notice that  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{5,3} = 21$ ;  $t_{1,1} = 23$ ;  $t_{1,0} = 2$  or  $3$ ;  $t_{2,0} = 1$ ;  $t_{5,4} = 19$ ;  $t_{3,1} = 20$ . Here the value of  $t_{3,0} \geq 4$ , which is not possible.

Let  $t_{5,4} = 20$  and  $t_{3,1} = 19$ . Notice that  $t_{5,0} = 26$ ;  $t_{5,1} = 25$ ;  $t_{5,2} = 24$ ;  $t_{5,3} = 21$ ;  $t_{1,1} = 23$ ;  $t_{1,0} = 2$  or  $3$ ;  $t_{2,0} = 1$ ;  $t_{5,4} = 19$ ;  $t_{3,1} = 20$ . Here the value of  $t_{3,0} \geq 4$ , which is not possible. Hence  $t_{5,3} \neq 21$ .

Similarly  $t_{5,3} \neq 22$  and  $t_{5,4} \neq 19$ ;  $t_{5,4} \neq 20$  therefore  $t_{1,0} \neq 2$  or  $3$ . Hence  $t_{5,2} \neq 24$ . Therefore  $t_{5,0} \neq 26$  and hence  $t_{5,1} \neq 25$ . We have proved that if  $t_{5,0} = 26$  the five star  $G = 4K_{1,2} \cup K_{1,13}$  does not admit a skolem mean labelling .

Similarly, we can prove the result for other values of  $t_{5,0}$ . Hence the five star

$$\begin{aligned} G &= K_{1,\ell} \cup K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n} \\ &= K_{1,2} \cup K_{1,2} \cup K_{1,2} \cup K_{1,2} \cup K_{1,13} \\ &= 4K_{1,2} \cup K_{1,13} \end{aligned}$$

is not a skolem mean graph. That is  $G$  is not a skolem mean graph if  $|m - n| = 5 + 3\ell$ .

In a similar way, we can prove that  $G = 4K_{1,2} \cup K_{1,14}$  is not a skolem mean graph if  $|m - n| = 6 + 3\ell$ . Hence on generalizing,  $G = K_{1,\ell} \cup K_{1,\ell} \cup K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$  is not a skolem mean graph if  $|m - n| > 4 + 3\ell$ .  $\square$

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By Franklin Roosevelt, an American President.

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[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.

[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

### **Research papers**

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

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June 2017

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